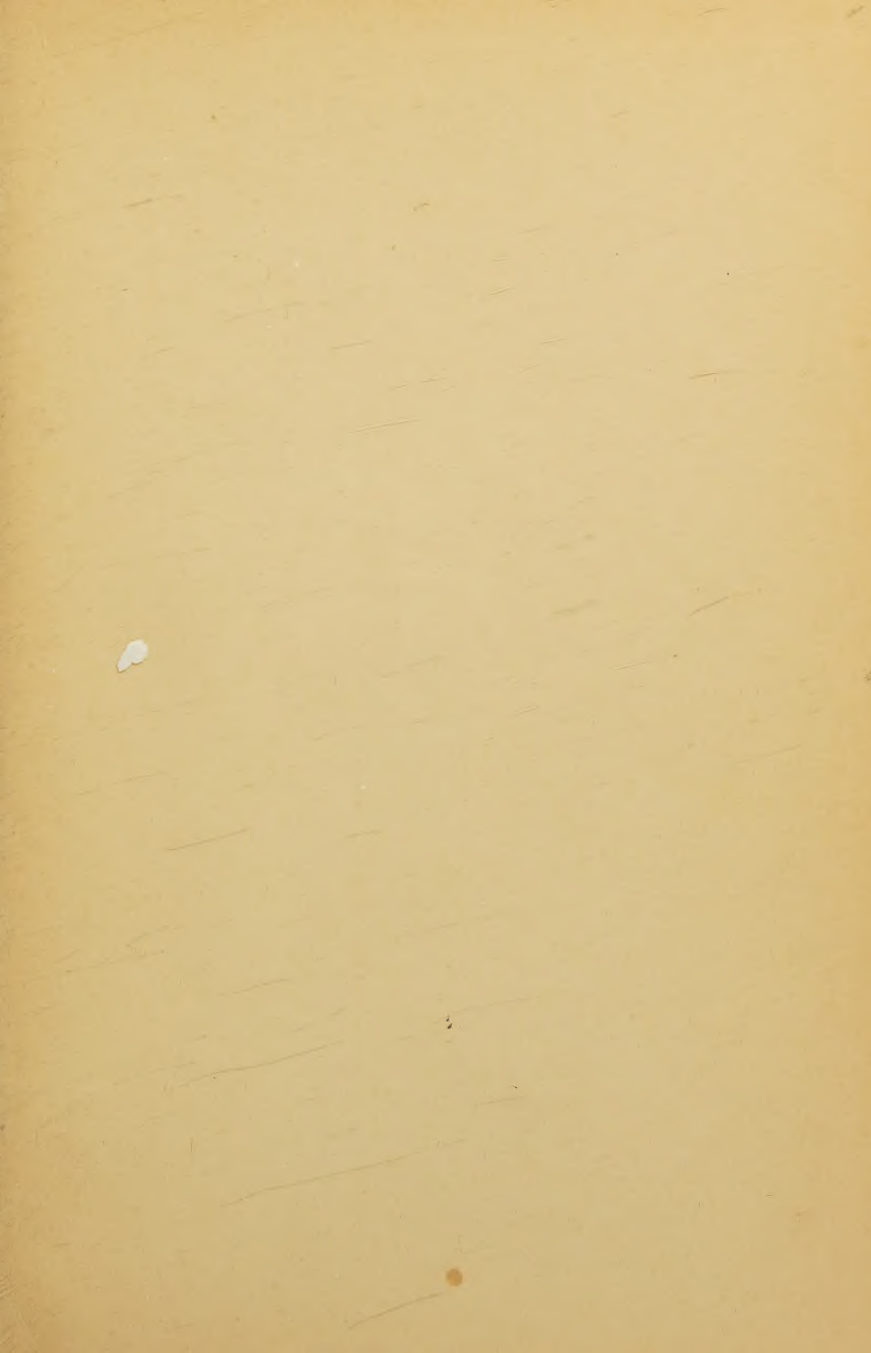






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THE TEACHING OF ELEMENTARY ALGEBRA



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WITHDRAWN

THE TEACHING OF ELEMENTARY ALGEBRA

BY

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WITH AN INTRODUCTION

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EDITOR'S INTRODUCTION

I SHOULD like to have every teacher of elementary algebra read this book; not only read it, but study and ponder it. There can be little doubt that elementary algebra, in its traditional form, has been the least satisfactory of our secondary-school mathematical courses. The author's analysis of the reasons therefor is interesting, his proposed remedy is at once original and sane. His discussion of the "practical" problem and of why it is unsuitable is illuminating. Above all, his method of problem analysis constitutes, I believe, a real contribution to mathematical methodology.

The author has brought to his work an unusual preparation. He has had many years' experience in shop and industrial practice; for the past several years he has taught in a public high school. He has thought deeply both on the need of the general citizen for mathematical training and on how to impart such training effectively. His fellow teachers may not in every instance agree with his conclusions, but I am confident that no thoughtful teacher will fail to find his exposition stimulating and suggestive. No further word of introduction on the part of the Editor seems called for. Let the book speak for itself.

J. W. YOUNG

PREFACE

ELEMENTARY ALGEBRA causes more trouble than all the other courses of the curriculum put together. It is usually held directly responsible for a high percentage of freshman mortality. Aside from this, the fortunate few who obtain passing grades seem unable to make much use of their training. They can solve the most complicated examples in factoring or simultaneous quadratics, but are unable to solve ordinary quantitative problems, which require insight and mental power rather than skill in symbolic manipulation. It is hardly necessary to add that most of them are totally unable to formulate a problem from scattered data or measurements; that is, to choose between the relevant and the irrelevant. In short, elementary algebra does not appear to be organized in such a way that tangible and practical results are obtained.

A large number of books claiming to remedy this situation have been published. We have all learned from past experience that books using purely deductive methods fail to produce satisfactory results with most students. Most of us have learned through bitter experience that purely practical books, which give only formulas and direction for their use in practical problems, do not contain that subtle and elusive thing which, once grasped, gives mastery over mathematical

processes. Some of us are now learning to our surprise that a combination of the former theoretical with the present practical also fails to develop sound habits of quantitative thinking.

In the past we have blamed, analyzed, and greatly improved various special features: training of teachers, textbooks, methods of presentation, and organization of subject, yet the situation has remained essentially unchanged. We must evidently turn our attention to the very foundation of the structure, the features that have stood so long that we have come to look upon them as being essential or unchangeable. The author proposes to discuss the following fundamental topics:

1. **Lack of proper motivation.** Motivating the subject is usually understood to mean the presentation of details and facts related to the subject in such a way that the pupil is induced to study. This narrow interpretation is one of the causes of our troubles. The author takes the position that a school or life task will be more expeditiously performed if the reason for its performance is plainly seen. Therefore an extensive discussion of the "practical" problem is included.

2. **The inductive method.** This method of teaching mathematics is very popular at the present time. The fallacy on which the method rests will be pointed out. This should not be taken to mean that a return to the old deductive methods is advocated. It means rather that a sane combination of the two methods, together with a clear understanding of the learning process,

seems to be the only sound foundation of mathematical teaching.

3. Relationships. What is meant by relationships among quantities? Can the student be taught to "express the relationships among the quantities in symbolic language," a common phrase in discussion, when the teacher does not show him how to express the relationships in plain English, does not teach him how to find them, often does not give a single illustration of what is meant by a relationship? A full answer to this troublesome question is given. A number of problems are analyzed, and as a further aid a simple diagrammatic way of exhibiting these relationships is given which will be found of great assistance in the analysis of problems.

4. The use of the function concept in unifying the subject. Every one speaks of the function concept as being the great unifying idea in mathematics. Yet few make use of it. It is the author's purpose to show that every problem solved by means of the function idea strengthens the pupil's grasp of the subject as a whole.

5. Fundamentals. The author recklessly plunges into this dangerous topic. He may be wrong in some of his conclusions, but hopes that the reader will be indulgent toward the mistakes of a pioneer. Let us stop talking about fundamentals in speeches and find out what they are. Let us drag what we call the fundamentals out into the open, examine them in

broad daylight, and see how they can be used to perform their main function, that of unifying the subject. What most people seem to call the fundamentals of mathematics are only the fundamental rules of symbolic manipulation.

6. *The characteristic formula.* This name is here given to that element in an algebraic problem which a little thought and use will show to be one of the true fundamentals of mathematics. It will be found to embody a unifying idea in problem-solving. It is an idea which has been functioning for ages, controlling mathematical thinking, yet apparently never clearly seen nor recognized. Without a clear conception of the rôle of the characteristic formula, it is difficult to see how to solve algebraic problems, excepting through the hit-or-miss methods in vogue among schoolboys and quite a few of their elders. Ignorance of the characteristic formula, therefore, is at the root of many of our troubles in elementary mathematics. No pains have been spared to make its working so clear that the teacher will find no difficulty in establishing sound habits in the recognition of quantities and their relationships, without which there cannot be any controlled association nor any effective quantitative reasoning. The idea of a characteristic formula is so simple, and its application so obvious, that the reader will probably pay the author the compliment of claiming that he knew it all the time. But when he sees that the *conscious* application of it is essential to the

long-sought general method for the solution of problems, as described in this book, he will agree that its teaching is of fundamental importance.

In addition, the reader will find some new methods of simplifying problems, of justifying formal drill and "academic" problems, of introducing symbolic notation and the concept of negative numbers, etc. The aim at all times is to present the material in such a way that it can be used by the teacher word for word, at least in the presentation of fundamentals.

No attempt has been made to discuss methods of teaching symbolic manipulation. The technique of developing skill therein is well understood and modern textbooks can hardly be improved upon in this field. Neither has any attempt been made to establish new methods in some of the advanced parts of the subject. After the child has made a sound start, has learned to organize the material that he learns, can solve problems, and feels that he is mastering a useful subject in the correct way, any standard method will produce satisfactory results.

The author wishes to render full acknowledgment to the faculty of the McClymonds High School of Oakland for their interest and aid in the way of criticism and suggestions after reading the manuscript. He is especially indebted to Principal W. A. Tenney, without whose assistance and continued interest the psychological experiments confirming the soundness of the methods described could not have been performed.

Most of all he wishes to express his appreciation to Professor J. W. Young, of Dartmouth College, whose constructive suggestions enabled him to organize the material on a sound basis.

PAUL LIGDA

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INTRODUCTION

THE average teacher finds it difficult to obtain satisfactory results. Methods of instruction which appeared easy to follow as expounded by some great educational authority are found to be applicable only after a class has been organized and well started, and only after it has a sound background of well-assimilated information. Great authorities too often forget to mention the insignificant details of the method through which fundamentals are mastered.

The same thing may be said of most textbooks. After the first few chapters are mastered, the rest is easy. But the trouble is that the contents of the first few chapters rarely are mastered; hence the rest is not easy. The first few chapters contain nothing which will give a grasp of the fundamental concepts and a view of the whole. Every author loves to get into the difficult part of his subject, which he would expound to students trained by some one else in the fundamentals. Authors are too prone to take for granted that the teacher has already learned what these fundamentals are and how to teach them.

Many teachers follow the lead of these authorities. They are impatient to get into the interesting parts of the subject, the parts which they assume will be interesting to their pupils. They do not like to linger on

the fundamentals, for a very sufficient reason: very few of them have themselves learned the fundamentals. They remember their own bewilderment when beginning the subject; they remember that after a while it became clear. They therefore assume that their pupils will react in the same way. Very few do so.

If we analyze honestly the results of our teaching at the end of the term, we find that about five per cent of our students have obtained a reasonable mastery of mathematical processes, mainly in symbolic manipulation. We grade these "excellent." The rest, including the most of those graded "satisfactory," have merely done their work in a more or less satisfactory manner. We feel that they will never be able to use mathematical methods with facility. Instructors in mathematics in any university will testify to the truth of this statement. Let us observe a diplomatic silence about those who "passed," and the failures.

It is our purpose to analyze the causes of this unsatisfactory situation and not only to suggest some remedies — the great national pastime in educational circles — but also to explain the "insignificant details" of successful methods used in *starting* the average class on its way to a real mastery of the subject. In short, our aim is to point out the stumbling blocks, and to help the teacher to remove them from the student's way. This will be done by:

1. Showing how the study of mathematics can be justified to the class.

2. Giving a clear and comprehensive view of the fundamentals.
3. Unifying the subject around the function concept, thus providing a definite method of attacking the problems met in elementary algebra.
4. Simplifying some of the traditional methods used in introducing various parts of the subject.
5. Showing the place of practical problems in the sequence of study.

These topics have been arranged with a view to classroom use. The reader, however, may begin with the chapter on "The Use of the Function Concept in the Solution of Problems" (Chapter VII). He will then obtain more quickly a clear idea of the plan followed.

THE TEACHING OF ELEMENTARY ALGEBRA



CHAPTER I

INTRODUCING MATHEMATICS TO THE CLASS

The situation in the classroom on the first day of the term

YOU have finished the routine details, the children are seated and attentive, expecting the traditional speech of introduction. What are you to say that will create the right attitude toward the subject? The first words, the first five minutes, will have a great influence on the results that you will obtain this term. You are about to create an attitude toward yourself and toward the subject which will in turn create good, bad, or indifferent habits.

The importance of these first few minutes can hardly be overemphasized. The word "curiosity" properly describes the feeling of the whole class. Here is a new teacher, in a strange place, talking about a strange subject. This subject has an evil reputation. Many children have failed in the study and have had to leave the school as a consequence. It is said by some to be as useless as it is difficult. What is the best way to

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obtain a good grade with the least work? These and similar thoughts are passing through their minds.

The teacher speaks: "Mathematics is the science of quantities . . . antiquity . . . the Phœnicians . . . the Arabs . . . the Hindoos . . ." and so on. The class listens more or less attentively, then forgets everything immediately on leaving the classroom. The ice has been broken, but not in the right way. The next day the struggle will begin. But it will be a struggle between the teacher and the children instead of being a struggle between the children led by their teacher and the subject.

What has happened? What should have been done? What is the correct way to interest the class, to motivate the subject? The answer to these questions forms the subject of this chapter.

Attitude of children

Let us return to the attitude of the children at the beginning of the teacher's introductory speech. It is an attitude of expectancy as well as of curiosity. Psychologists tell us that curiosity can be changed to motivating interest, while expectancy leads to thought. Unfortunately, they are rather hazy as to the means through which such desirable results can be obtained. They have told us that we must arouse and maintain interest, but have failed to describe a clean-cut and result-producing plan of action. The principal reason for this failure is that the nature of interest is usually

misunderstood. This is well proved by various educational fads which appear from time to time, all justifying themselves by the claim that they arouse the interest which all other methods have failed to arouse.

Analysis of interest

In the final analysis, interest in most instances is only a product of egotism. We are interested in persons, objects, and subjects in the exact proportion in which they promise to influence our immediate or future welfare or environment. Hence, if we do not perceive, or cannot be made to perceive, any relation between them and ourselves, our interest will not survive the initial stage of novelty. Such interest deserves only the name of curiosity. It is a mild and polite feeling, not the motivating interest that induces effort. What interests people? "Themselves," answered Lord Northcliffe. He understood the fundamental psychology of interest.

This suggests a plan of action. We must relate the subject to the students. This is usually done by relating it to life in general, by showing that some one uses it somewhere. Then it is fondly expected that the students themselves will somehow make the rest of the connections. A few of the more intelligent do so, but the great majority do not. This is due to the lack of realization on the part of teacher or author that the background in youth is very limited, and that the relations given are not numerous enough, not suffi-

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ciently emphasized, nor understood clearly enough to create a lasting impression. Each student must be convinced that the subject has an important bearing on his own future welfare. Besides this, he must frequently be made to recall this conviction until the right attitude is created. That mild fear of the future which is usually called "ambition" must be aroused and fed by direct or indirect means.

The seven questions

Psychologists have formulated laws of learning. Unfortunately they have not yet given us laws that can be used to advantage by an average teacher, when trying to persuade an average, skeptical, and unwilling student that mathematics or indeed any mental effort is good for him. The following home-made analysis may lead us to the formulation of such laws. It describes the usual reaction on the part of the pupil when a new subject or object is introduced.

1. *What is this?*
2. *What is it good for? (Who uses it?)*
3. *Will I ever have any use for it? Show me.*
4. *How does it work?*
5. *Will you let me try it?*
6. *Why does it act this way? (What are the underlying principles?)*
7. *What other uses has it? (Generalization.)*

Questions 1 and 2 are due to pure curiosity; judgment is suspended. It should be noted at once that

a satisfactory reply to these questions must give the pupil a conception of mathematics *as a whole*, must give him some idea, however vague, of a general plan. The exhibition merely of details is fatal to future interest. We shall return to this point presently.

Until you have answered Question 3 in such a way that the student is absolutely convinced that he needs mathematics, you cannot secure his best efforts. He is probably a descendant of Euclid's practical pupil. The subject does not promise to influence his life. He will not in general be induced to ask Questions 4 and 5 because he is not interested in them. To expect Questions 6 and 7 under such circumstances is futile.

How to make each individual in the class realize that he actually needs the subject for immediate or future use is then the teacher's first and probably most important problem, because once this need is realized the class will practically teach itself. Incidentally this will also solve the discipline problem.

A crude attempt to obtain this result is usually made when the students are told, in indefinite terms, that the subject is important, or when vague reasons are given tending to show that their efforts will be rewarded in the dim future. But such talks are useless; for, as Dewey says, "General appeals to a child to think (and hence to study), irrespective of the existence of some difficulty that troubles him and disturbs his equilibrium, are as futile as advice to lift himself by his bootstraps." The average boy, like the average man,

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wants action. He is impatient of results. Also he has only average intelligence; that is, unlike exceptional students, he fails to perceive from a few casually spoken words that there is any relation between the subject and himself. He has often heard similar speeches and never noticed any startling benefits to himself due to hard study. In short, he fails to react in the desired way, and may even react negatively since the word "study" directly and at once suggests mental effort, in most cases a painful process!

Let us, then, begin by answering Questions 1, 2, and 3 in the best possible way before we begin the actual study of mathematical processes, principles, notions, and devices. Questions 4 and 5 will be answered later in the present discussion; for answers to Questions 6 and 7 the reader is referred to any standard textbook on mathematics.

Question 1: What is mathematics?

Classroom presentation. Mathematics is a tool. Through the use of it we obtain answers to questions. To what kind of question? To questions containing such expressions as *how long*, *how much*, *how heavy*, *how safe*, *how fast*, etc. Such questions are often asked when direct measurements with tapes, scales, gallon measures, and similar implements are not possible or practicable. These questions, called "problems," usually contain numbers and quantities and certain relations between them, some known and others unknown.

These quantities and relations must be manipulated according to certain definite rules before the answers can be obtained. The study of mathematics consists mainly of the study of these rules and of the devices through which they are applied to the solution of problems. In short, mathematics is a tool. We use it to obtain answers to questions regarding quantities.

A quantity is anything that can be measured, anything that can be increased or decreased, anything that can be expressed in numbers, in terms of units. We measure quantities because our daily work is essentially concerned with quantities. Lengths, volumes, weights, and costs are continually affecting our lives and fortunes. We must learn to calculate and manipulate quantities. We must learn to use the tool by means of which this is done, namely, mathematics.

Discussion. It has been found by actual trial that an answer of this type satisfies the children. It is certainly far more satisfactory than the following types of answer:

1. Mathematics is the science of quantities, or of numbers.
2. It saves time to use letters instead of words or numbers.
3. Mathematics is a generalized arithmetic.

Our answer appeals to practical-minded children. They are ready for the next question.

Question 2 : Of what use is mathematics? Who uses it?

Classroom presentation. A large number of problems in our textbook illustrate the practical use of mathematics in shops, stores, and construction work, mainly in finding costs of products and processes. Besides technically trained *scientists, teachers, inventors, engineers, and surveyors*, we find that the following classes of men have a special need of mathematics:

(1) *Farmers and business men* who want to know costs before ordering material, machinery, or buildings. They must know selling costs, overhead charges, and profits. They need and demand exact figures in detail for every step in transportation, manufacturing, and distribution. They must be skillful in detecting those errors the neglect of which sometimes causes large losses.

(2) *Contractors and managers of plants* who have to bid on contracts. They must know exact costs, weights, strengths, dimensions, and sizes.

(3) *Estimators, assistants to managers*, employed by the members of Classes (1) and (2).

(4) *Draftsmen and designers* of machinery, buildings, etc.

(5) *Foremen*, who have to calculate the small detail work constantly arising in shop routine which the untrained mechanics cannot handle.

(6) *Skilled mechanics* who have to lay out the work for the machine hands and routine workers.

Competition is becoming more and more strenuous.

Rule-of-thumb, snap judgment, guessing methods are not allowed in important work any longer. These men must be sure that they are right before they do anything. A mistake or a wrong guess usually means loss of life or property. Everything, as far as is possible, must be foreseen or calculated accurately and efficiently. The person who does not know how to plan and calculate size or cost is of little use in important positions. The mechanic works with hand tools; the man who directs the mechanic's efforts uses the brain tool, mathematics.

We live in an age of rules. Whenever we wish to do anything without knowing just how to proceed, we ask advice from some one who does know. This person usually gives us rules: First you do this, then do that, and so on, until you obtain the desired result. Whether you wish to play ball, solder a kettle, wind a dynamo, or build a house, you first make sure that you know the rules that you are going to apply. Arranged properly, they form your plan of action. They are the result of the experience of mankind. They have proved their value in numberless cases. To disregard them may be the means of discovering some better way, but more usually leads to trouble and loss.

We all use rules. In fact, the more rules we know and can apply, the higher our skill and standing. Some of the rules relate to manual work, the operation of tools and machinery. Still other rules relate to design, measurement, and laying-out. At present we are concerned with these latter rules only.

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Measurements are of two kinds, direct and indirect. Direct measurements are obtained by the use of such implements as yardsticks, protractors, scales, etc. The use of these is well known or easily learned, and further mention is unnecessary.

Indirect measurements come into play when we wish to measure any quantity to which direct measuring tools, for various reasons, cannot be applied. Thus the object to be measured may be impossible to reach; it may be inconvenient to handle; it may even not exist, except in our imagination, as when we design.

The tool used for indirect measurements is mathematics. The only alternative is the guess.

Discussion. The above statements about people who use mathematics usually produce startling results in the class. The pupils are fully prepared to hear that they will need mathematics the moment they begin to work in shops, offices, or stores. But they have already learned from various sources that almost no mathematics is actually used by the average person, so the teacher's statement would have produced very little result. They are prepared for appeals to exert themselves on the ground that the subject will be immediately useful. But they are totally unprepared for the statement that they will not need mathematics so long as they are satisfied to remain in lowly positions. In this unprepared state the suggestion that only better-salaried persons need mathematics, with the accompanying fear that they may not be among the

elect unless they exert themselves, enters their minds. This is the motivating thought. It causes the student to answer the third question himself, the best and most convincing answer.

The ethical purist may take issue with this method on the ground that mercenary motives should not be used, and that the only appeal to the lower instincts ethically permissible is inducing fear of failure in promotion and graduation. The same purist speaks of rousing ambition without realizing that ambition is partly fear of not having a good position in the future, and, also partly, a desire to have more than one's share of the good things of life. The only answer needed is that appeals to higher and better motives made to adolescents have very seldom produced the desired reactions. It is about time for educators to begin to realize this fact.

Let us speak plainly. There is little use in denying that the average child is intensely selfish and practical-minded, as well as complacent and perfectly satisfied with what he knows. He must be made to realize that training for life, like training for athletics, is a series of strenuous efforts; that men who occupy responsible positions have put themselves through intensive training for those positions. The child must be shaken out of his complacency. His ambition must be aroused by a specific and direct method. It must be motivating ambition, not merely the vague desires that usually masquerade under the name of ambition.

Other uses of mathematics

While the above method will produce good results with the majority of pupils, it will fail with quite a few, especially with girls. The latter can hardly be expected to take much interest in industrial efficiency. For their edification, then, the teacher should mention other places where mathematics is used:

1. In the study of sciences and in general scientific reading.
2. In keeping household and personal accounts and budgets.
3. In household management and the care and training of children.
4. In assisting children with their studies, by parents.
5. In forming valuable habits of planning and logical thinking, thus directly useful in language work.
6. In the arts: music, sculpture, painting, and design.

The teacher should have such arguments at his ready disposal, since doubts of the usefulness of mathematics are hard to dispel. Yet they must be dispelled if earnest effort is desired. Each of these doubting Thomases must be convinced that the subject is useful to him. No real interest or progress can otherwise be expected from any but the exceptionally intelligent few who find mathematics an enjoyable intellectual recreation.

Most textbooks do but little to assist the teacher in this vital point. Authors seem to consider that the beauty of logical reasoning, a few mathematical puzzles solved in a way novel to the pupil, and a few problems with a "life setting," will vitalize the subject. Some authors even go so far as to state that mathematics has only a disciplinary value; that all the mathematics that people need in life-work can be taught to them in a few lessons. Pedantic statements of this kind, unsupported by any specific proof and displaying gross ignorance of industrial activities and needs and of the length of time required to assimilate fundamental principles, are gravely quoted and too often form the bases on which curricula are planned and textbooks on "practical" mathematics are written.

The need of mathematics

There are plenty of mathematical problems in life-work, but they generally remain unformulated owing to ignorance. Often they are not even recognized, somewhat as danger is not recognized by little children. It is natural in such circumstances to claim that they do not exist! The need of mathematics is then more real than apparent. Unable to avail themselves of mathematics, people substitute guessing methods with all the attached danger and costs. Then to the end of life they remain convinced that mathematics is a useless or unnecessary subject. Breakages, accidents,

losses of contracts, and so on, are considered inevitable factors in business. Not having standards of comparison, they never realize the high cost of their ignorance.¹

Let us now assume that the student is thoroughly aroused and in a responsive frame of mind; that he is convinced that he needs mathematics. He is ready to do a reasonable amount of work, provided that he sees some plan of work leading to a definite result, and provided that we remind him from time to time that mathematics is useful and practical. We will discuss the plan, show how the child may be made aware of it, then examine the methods that should be used in constantly recalling his initial impressions; that is, in sustaining his interest.

¹ For a further elaboration of this point see Chapter IX.

CHAPTER II

FUNDAMENTALS OF MATHEMATICS — THEIR FUNCTION IN THE LEARNING PROCESS

Question 4: How does it work?

THE student is now ready to give careful attention to an exhibition of mathematics in action. His attitude can be compared to that of a customer convinced by a salesman that a machine is worthy of a careful trial. The trial begins. The customer watches the machine, first as handled by an expert, then by himself. The fact that he does not get as good results as the expert does not discourage him. He is willing to submit to a period of learning. He has faith in the machine, for he has seen it producing good results. He acquires skill in operating the machine by following a definite plan. He runs it as well as he can, then studies the parts, runs it again, again examines the results, and repeats this until he thoroughly understands the operation.

The teacher who is selling mathematics to his class must create a similar situation. He must first show how mathematics is used. He takes a practical problem which the student cannot solve, although the latter fully understands the situation described and the need for a solution. The teacher analyzes the problem and through the manipulation of certain symbols obtains a solution. The student is led to perceive that this manipulation follows some definite law of procedure.

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He sees mathematics working as a machine. He sees the useful results and is eager to try his skill. He understands that he must submit to a period of learning, because he realizes that he must learn something of the parts before he can operate the whole. *Yet he is aware of the existence of the whole.* He sees dimly that there is a general plan and that each part has some definite relation to the whole and is necessary thereto. The following example, illustrating the use and value of mathematics, has been found to be very effective at the beginning of the term:

Classroom presentation. A plumber was offered 65 pounds of solder of unknown composition. If it contained a reasonable amount of tin, it was a bargain, tin being worth 30 cents a pound, while lead is only worth 10 cents. In order to ascertain its composition, he weighed the mass in water and found that it lost 6.025 pounds of its weight. If tin loses 0.137 of its weight in water, and lead loses 0.089, how did he find the weight of the tin in the mass? (Specific gravity of tin = 7.35; S.G. of lead = 11.4 approx.)

We can make this statement (write on the black-board):

The tin and the lead weigh 65 pounds.

In symbols:

$$T + L = 65$$

or,

$$T + L = 65 \quad (1)$$

The weight lost in water by one pound of tin is 0.137. The weight lost by T pounds would be $0.137 T$. Similarly the weight lost by L pounds of lead would be $0.089 L$.

We can then make this statement (write):

Weight lost by tin added to weight lost by lead is 6.025 lbs.

In symbols:

$$0.137 T + 0.089 L = 6.025$$

or,

$$0.137 T + 0.089 L = 6.025 \quad (2)$$

Multiplying each side of (1) by 0.089, we have

$$0.089 T + 0.089 L = 5.785$$

Subtracting this from (2) we obtain:

$$0.048 T = 0.24$$

Dividing each side of the equation by 0.048, we obtain $T = 5$. That is, there were 5 pounds of tin in the solder. From this result the plumber calculated its true value.

Discussion. The objection that problems and methods of this type discourage the students is found to be unsound in most cases. The students constantly see difficult work, utterly beyond their own small skill, performed by experts in various lines. We usually find that children are stimulated rather than discouraged by such experiences. Difficult feats challenge

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them to effort, because they realize that they may be required to duplicate them. On the other hand easy tasks fail to interest them, for they know that they can perform them if necessary, without training.

The above method of presentation, followed by practice drills, should be repeated frequently. The presentation should not be only of material that the student can learn thoroughly or use immediately. This would defeat the main purpose, which is *to give some conception, no matter how vague, of the subject as a whole*; of how the mathematical machine works in the hands of an expert. This conception of the plan will become stronger with each repetition, while the parts remain unified and interrelated. The subject will not be built piecemeal. It will gradually emerge into the full light. It will advance from the vague to the definite.

At the beginning of the term, then, illustrative examples should be frequently presented. They should be fully worked out by the teacher. A wealth of detail should be put in, with frequent references to the unifying plan. Besides a conception of the subject as a whole, the aim should be to create familiarity with the vocabulary of mathematics. A number of problems in this book fully illustrate the method of procedure.

Fundamentals

The word "fundamental" has been much overworked and misused. Everybody speaks of funda-

mentals; but few define them, state them, or show the use of them. Each mathematician has his own list of fundamentals, differing from other people's lists. The author takes advantage of this custom to give his own list. This will be done after what may be a rather novel definition of fundamentals is offered.

The fundamentals in any science are the principles, facts, and concepts which the student must know well and understand clearly before beginning the study of that science. We do not establish fundamentals; we connect the subject to them. When the principles, etc., of the science have been connected with what the learner already knows, then, and only then, do they in turn become the fundamentals. A proof of this is easily given. If we have not learned the principles called fundamentals by the authors it is because we have not associated them when presented. Associated with what? Obviously with what we knew before we began the study. Thus the axioms of mathematics become fundamentals as soon as the child perceives that what he already knows is utilized in the new subject through the use of axioms. In short, establishing fundamentals means that we show the child that he already knows something about the subject. We should organize what he knows before introducing new facts.

Too often it is forgotten that in every subject there are only a few main ideas, while there are many details and facts. The latter become significant and useful only after they have been related to these main ideas,

and through them to each other. The whole system of ideas then functions as a unit. These main ideas must be acquired and defined through their relation with what previously was fundamentally known, preferably through some concrete experience. This fundamental relation must be established as early as possible; these main ideas must ever be kept in the foreground; and new material must consciously and deliberately be related to them.

The fallacy of the inductive method

The author fully realizes that the above plan runs contrary to the present popular, so-called psychological, method of teaching, which begins with new details, particulars, and concrete elements. This latter method was developed in reaction against the excessive difficulty of the older, so-called deductive, method. Like most reactions it went to the other extreme, creating evils which are just as great as those it avoided.

The main claim of the partisans of the inductive method is that children learn details with more ease and interest through it than through the old method, and that therefore it is the correct psychological method. This is perfectly true so far as the details are concerned; learning details is the psychologically correct method of learning details. But that is all. In practice we find that things learned psychologically are forgotten just as psychologically, that is, just as fast and easily. After learning what seem to him unrelated

details by associating them with other unrelated details the child fails to organize them into a significant whole which functions as a whole and which he can use. He fails to recall the details when he needs them because they have been associated during the learning process with other disconnected details. The teacher or author has failed to show him how to organize his knowledge. Complaints from university instructors and from business men show the results of the inductive method. "Young people do not know anything well and are not interested in anything worth while," is a well-nigh universal criticism. The inability of many people to take any interest in broad questions, to foresee consequences of acts and policies, may also be attributed to this vicious method of instruction. How can we expect people to grapple with great problems when we teach them only details, and carefully restrain any attempt at a generalization "before the pupil is prepared for it"?

Some never are prepared for it. They are deliberately left without any guiding plan for a whole term. They are then expected to develop organizing ability unconsciously, without practice or help. Our inductive friends forget that very few people have the maturity of mind necessary for unaided generalization — the power to deduce a principle from a large number of apparently disconnected details.

Reason for its popularity

The inductive method is popular because it is easier to teach details and parts separately than it is to teach them in an organized way. It is easier to compile catalogues of mathematical processes and facts, claiming that the contents are arranged psychologically, than it is to write result-producing textbooks: the kind that organize facts as fast as they are presented. The inductive method is also popular because it yields immediate though small results. It looks like a short cut to knowledge. It appeals to our sense of efficiency and to the general human desire to get something for nothing. But we are beginning to notice that we are not obtaining lasting results, and hence to suspect that the method must be fundamentally wrong.

Common-sense tells us that we cannot begin by examining the means to the end before we ascertain what the end is. A child cannot build out of parts a whole that he has never perceived. To expect him to discover a relation between a fact and a unifying principle or definition, in the total absence of the factor to be related, may be psychologically correct; but has any child ever lived up to that expectation?

“Thought consists in the establishment of relations. There can be no relation established, and therefore no thought framed when one of the related terms is absent from consciousness.” (Herbert Spencer.) How does this agree with “Definitions and general principles must grow out of the pupil’s concrete experience”?

They must, but they never do! They cannot grow unless the seed is present. The outlines of the definitions and general principles must be present from the beginning of the study, and they must grow more definite as progress is made. It is not necessary nor is it possible for the child to understand or see them clearly from the first. They may be dim, but the mind vaguely perceives some relation to each detail learned. Constant recall, however, will eventually make them more significant.

The abuse of the deductive method

The deductive method was sound in one point. It started from a unifying principle or definition, and worked down to the details and applications. A dim perception of the whole was obtained, together with its relation to the details. So far, the method was producing good results. Repeating the process a few times with different details would give the students a fair idea of the function of the principle or definition. But some overzealous logicians, the efficiency experts of the period, began to tinker with the machine. They decided that it would save time to learn thoroughly all the fundamentals at the beginning of the course. The students were then required to memorize them and to have them ready at the tips of their tongues whenever their applications were needed. They were obliged to recite them and explain them clearly before they ever had a chance to apply them; for the applications came

after the essentials were "learned." These essentials were also made criticism-proof by producing "gems of human thought," much admired by logicians, but incomprehensible to children, owing to the kind of words used and to the length of all-embracing sentences. We all know the results of this extreme formalism. The public rebelled and the whole method was scrapped, the good along with the bad. A thorough reform was demanded.

The abuse of the inductive method

Radical reformers rushed at once to the other extreme. They started their courses with concrete facts and details. So far their method was sound. If we present something known and tangible before we present the unknown we stand a good chance of forming the right association with an organized fact, and hence of getting the unknown learned. But the efficiency fanatics who started from the right premise: "Knowledge rests on concrete experiences," arrived at the wrong conclusion: "Therefore imparting knowledge must consist of the presentation of concrete experiences only." They said that all true knowledge begins with concrete experiences and ends the same way. This may be true, but what happens between the beginning and the end?

These short-sighted reformers tried to improve educational methods by presenting concrete elements by the cartload, and expected the pupils to classify and

generalize the entire mass at once at the end of the period, or worse still, at the end of the term. This was to be done by the unaided efforts of immature pupils who were to develop the power of generalization "psychologically" out of thin air, without any practice or even consciousness of the process. Such feats are sometimes performed by exceptional students, due to unconsciously correct generalization and classification developed in spite of the method. Such students are then cited as examples of the soundness of the procedure.

Illustrations of such methods are given by textbooks of so-called "practical" mathematics. These try to teach the use of formulas before the student has acquired the fundamental notion that the various parts of geometric figures are related to each other. Similar methods are exhibited in modern "practical" arithmetics when children are trained to juggle numbers before forming judgments of relative magnitude or are expected to solve problems without the fundamental notion that quantities are related in definite ways.

It must be acknowledged that children like the inductive method. To them it seems practical and useful, as it apparently shows quick results. It is easy to work exercises that follow a given model. But when an attempt is made to increase the difficulty, its attractiveness disappears. It requires but little mental effort to memorize predigested facts and rules which can be recalled by any one who knows the cues. But

the only persons who know these cues are the authors or the teachers. The result is that the children can do fine work in class or with some particular book, and still fail to recall anything when the problem is presented in some unfamiliar setting, outside of the classroom. We are beginning to hear persistent talk about soft pedagogy. The pendulum is swinging back to the deductive method. Let us try to stop it halfway between the two.

Analysis of the learning process

A comparison of the good points in both the inductive and deductive methods leads to an analysis of the learning process somewhat as follows:

The mind vaguely grasps something as a whole, moves from this to some of the parts, returns to the whole which then becomes clearer, returns to the parts embracing a few more, and so on. This may be called an inductive-deductive, analytic-synthetic, or concrete abstract process. The mind combines a number of parts into a whole and separates it again into parts, repeating the process so long as further details are discovered to make up the whole.

Objections to the natural way of learning

The above describes the natural way of learning. But looking at it from the point of view of modern needs, this natural way is defective in one vital point: the "whole" described in the preceding paragraph may

be anything plausible in appearance and available at the time of the presentation of the unknown material. That is, unless properly directed, we may organize and classify with the wrong group. Thus we may see a bat flying and by a natural error classify it with the bird group.

The second objection to this natural way of learning is that undirected and haphazard classification does not provide any means of recalling the facts learned. But modern education has a definite objective: certain facts must be learned and they must be under control; i.e., they must be recalled whenever needed. Obviously then we must introduce the right "whole" in the learning process, the function of this whole being to organize groups of facts and to serve as a beacon guiding the child's effort.

We find also considerable misunderstanding about the "known," to which the new fact is to be related. Teachers and authors are inclined to consider as known anything which they have at any time mentioned, or that the children can "recite." They confuse the "known" with the "should be known." They also consider that anything which they term a fundamental immediately becomes one, and that whole chapters of the subject can be related thereto through their statements. They can be so related, but the connecting bonds are mere threads, not cables that can stand the recalling strain. Every teacher can tell of the extraordinary misconceptions of fundamentally im-

portant principles revealed in examination papers or in problem work as a result of such teaching.

An illustration

The results obtained by the inductive and deductive methods may well be compared by taking a hypothetical case. Let us assume that it is purposed to teach a class how to set up and operate a linotype machine. Assume also that this class knows nothing about linotypes or their use, and cares still less.

A teacher using the purely inductive method would bring a number of separate parts such as gears, keys, shafts, and pinions to the class. He would have his pupils learn the names and uses of these parts by comparison with similar parts of well-known machinery such as automobiles or watches. At the end of this course the children would be expected to know how to set up and operate a linotype. They would be expected to discover by themselves what the whole machine looks like and how it is used and operated. For have they not learned the parts, more quickly than they could by any other method?

On the other hand, a teacher using the purely deductive method would show blueprints of a linotype on the first day and briefly explain its use. He would then state those fundamental principles of mechanics and chemistry involved in its design and use, requiring that these be memorized. The children would be instructed to learn the application of these principles

to the various parts, class work being done from blueprints. We can imagine what amount of interest would be displayed! At the end of the course the children would be expected to be able to learn actual operation of a linotype in minimum time.

Obviously neither method would produce satisfactory results. The sensible way would be to show the machine in operation at intervals during the course, to use blueprints and separate parts, and to study the parts, not only as independent units, but also with due regard to their position in the machine, their function, and their relation to the other parts, and to teach the principles involved. At the end of such a course the children would be ready to begin intelligent study of the installation and operation.

This third method apparently requires more time than the other two. In practice, however, it would be found to require less time than the first method because it is easier to learn parts when the general plan is ever present, to which every part may be consistently connected as well as to the other parts. It would also be found to require less time than the second method because interest is maintained by actual handling of the parts and by sight of the machine in operation.

The third method clearly follows the natural method of learning. First we consider the whole, then some details, the whole again, then other details, and so on. The machine gradually becomes better known as the study progresses, not through addition of detached

parts, but as a complete unit. From the beginning of the course to the very end the learner never loses sight of the whole plan, of the use of the machine, of the method of operation, nor of the purpose of the course, that he will operate the machine eventually.

Application of the third method to the study of mathematics

At the beginning of this chapter we learned how to show the use of mathematics. The illustrative examples throughout the book aim to maintain interest, that is, to keep in touch with the purpose of the study. A conception of the general plan and methods of operation must gradually be acquired by the pupil through the constant use of the tools that we call the fundamentals. The two main functions of fundamentals are: (1) our authority for statements, definitions, principles, and processes; (2) our unifiers and organizers of the subject.

It is not essential thoroughly to understand every possible meaning and application of a fundamental before using it for the above two purposes, any more than it is necessary to know every possible use of a hand tool before using it for some specific purpose. Lack of realization of this fact has kept many teachers from organizing their courses in a desirable way. They have felt that they ought to do everything "thoroughly."

Fundamental notions essential to learning

We may take for granted that the following conditions essential to learning have been established.

There must be present:

1. A personal motive, creating the right attitude toward the subject.
2. Some general notion of the end in view.
3. Some unifying principle, more or less clearly understood.
4. Some known fact with which association is made.
This fact must be known to be related in some way to the unifying principle, or fundamental.

Keeping in mind these four conditions, the author offers the following list of fundamentals which must be introduced and assimilated as early as possible. They are the means by which the teacher will lead the child to a mastery of mathematical processes.

1. I want to learn all I can about mathematics.
2. A problem is not a meaningless puzzle. It can be shown to be in most cases an abbreviated description of some difficult life situation which some person has had to meet, and which I in turn may have to meet.
3. Higher mathematics is used in the solution of problems when arithmetic methods are inadequate. Its processes are easier, quicker, and less liable to error. The results obtained are easier to check.
4. Statements of problems contain or imply some

kind of relationship between the quantities involved. (This is the function concept in its initial form.) Such relations can be expressed by means of sentences. These sentences can be translated into equations.

5. The general plan of solution of a problem is then:

- (a) Recognize the quantities described and implied in the statement of the problem,
- (b) List the statements explicitly made concerning these quantities,
- (c) Discover the form of the characteristic formulas ¹ needed,
- (d) Translate these statements and relations into symbols and solve the resulting equations.

6. Symbols stand for words and sentences. They are used because they show at a glance the relationships between the quantities, the plan of operation, and the necessary processes. It saves time to use symbols, and they are easier to handle than sentences.

7. The aim of symbolic manipulation is to place one unknown on one side of the equation alone, all the other (known) terms being on the other side. To attain this end we perform operations not contrary to the law of equations: namely, whatever operation is performed upon one side of the

¹ This term is fully explained in Chapter VII.

equation must also be performed upon the other side.

These notions should constantly be reiterated by the teacher. They are the motivating and unifying principles.

Laws of symbolic manipulation

We have also fundamental rules or laws of symbolic manipulation. Their main function is to authorize certain special processes. They are well known and we will mention them without comment, merely calling attention to the advantage of a brief and simple preliminary statement or definition, even though it may not be completely accurate. These laws are:

1. THE LAW OF EQUATIONS: *Whatever is done to one side of an equation must also be done to the other side.*
2. THE LAW OF SIGNS: *Either in multiplying or dividing, like signs give plus, and unlike signs give minus.*
3. THE SUBTRACTION LAW: *To subtract, change the signs and add.*
4. THE EXPONENT LAW: *When two powers to the same base are to be multiplied, the exponent of the product is equal to the sum of the exponents; when they are to be divided, the exponent of the quotient is equal to the difference of the exponents.*
5. TO REMOVE PARENTHESES PRECEDED BY A MINUS SIGN: *Change all the signs within the parentheses.*

Special fundamental devices and processes

1. Graphs and tabular forms.
2. Characteristic formulas in their various forms and aspects.
3. Linear equations.
4. Simultaneous equations. The principle of substitution.
5. Quadratic equations.
6. Various minor principles, rules, and devices.

Most of the failures in the study of mathematics can be traced to the neglect of the "fundamental notions" by the teacher or by the textbook in use. Until the student has acquired at least a few of these notions, no motivating interest can reasonably be expected from him, no matter how well the laws are illustrated or the fundamental devices are introduced and presented. He sees no reason for exerting himself.

CHAPTER III

THE INTRODUCTION OF SYMBOLIC NOTATION AND ITS USE

The meaning and use of symbols

BEGINNERS find great difficulty in realizing that the operating signs $-$, $+$, $\times$, $\div$, etc., can be used to represent many different words and meanings in various problems. In their case operating symbols are to a great extent abstract, and consequently of little assistance in the solution of problems. Beginners also fail to realize that the letters used represent numbers. For these reasons our next discussion will aim to make clear the use of symbols as well as to show the use of mathematics.

Classroom presentation. My purpose to-day is to show one of the methods used in the solution of problems. We may call it the *symbolic method*.

A symbol is a sign which stands for a number, a word, or a thought. We use symbols because they make our work easier and save space. Thus, instead of writing one hundred and thirty-seven dollars and sixty-five cents, we write \$137.65. If we were told to multiply seven hundred and sixty-two by three hundred and fifty-one, we would write it this way:

$$\begin{array}{r} 762 \\ \times 351 \\ \hline \end{array}$$

The figures 762 and 351 stand for the words seven hundred and sixty-two and three hundred and fifty-one respectively, and the sign $\times$ stands for the words *multiply by*. We took our written directions and translated them into symbols. Why? Because it is easy to multiply with figures, while it would be very difficult to multiply if we had to use words only. The result of the multiplication is also obtained in figure symbols which can be translated back into words whenever we choose.

Problem-solving in mathematics is performed in the same way. We take word-problems, we assign symbols to the quantities and relations described. Then we manipulate the symbols so as to obtain the result that we desire. By *quantity* is meant anything that can be measured, anything that can be increased or decreased, such as length, period of time, area, volume, cost, etc.

When we have a problem to solve, we first resolve what is to be done; we form a plan of action. After this we apply such rules and processes as we know to what is given: the numbers, relations, and quantities. When we cannot solve a problem, it is always because we cannot see our way clearly; we do not know where to begin and what to do. That is, we have no definite plan of action.

You should keep in mind the fact that every problem contains the suggestion of the plan of solution. Unfortunately, this suggestion is often hard

to find, being mixed up with other words and sentences.

In order to find it, it is usually advisable to remove this unnecessary matter, to husk the problem. After this you can rewrite it in fewer or different words that are more significant.

Example. From a cylindrical tank, half full of crude oil, 100 gallons are removed and 36 gallons are lost by evaporation and leakage. The tank is then one third full. How much does it hold when full?

You should always read the problem carefully and try to guess the probable answer. This is very important, because you can usually guess fairly close to the right answer, thus getting a check on your work in case you should obtain some absurd result through some mistake in calculation. Besides, you will thus learn exactly how much you can trust to your guessing ability. The tank looks as if it would hold about 800 gallons.

On re-reading the problem, you decide that the words *cylindrical, crude oil, by evaporation and leakage*, are not necessary to the solution. They distract your attention from the main points. Also you can combine the amounts removed: 100 and 36. You then rewrite:

From a tank one half full, 136 gallons are removed, leaving it one third full. How much does it hold when full?

It is clear that you obtain a better view of the pro-

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blem when it is in this form. The solution is easier because you can see some relation between the quantities. But suppose that you cannot see any plan of solution. In this case change the rewritten problem:

From one half of the tank 136 gallons are removed, leaving one third.

Or, in a different way, using figure symbols:

From $\frac{1}{2}$ tank we take away 136, the result is $\frac{1}{3}$ tank;
or,

$\frac{1}{2}$ tank less 136 equals $\frac{1}{3}$ tank;

or,

$\frac{1}{2} t$ $-$ 136 $=$ $\frac{1}{3} t$.

If you let the symbol t stand for the number of gallons in the full tank, the symbol $-$ stand for the words *less*, or *take away*, or *are removed*, the symbol $=$ stand for the words *equal*, *is the same as*, *leaving*, or *the result is*, you have the problem in this form:

$$\frac{1}{2} t - 136 = \frac{1}{3} t.$$

The plan of solution is now obvious. The difference between one half and one third is one sixth. This sixth is equal to 136 gallons. Hence the whole, or six sixths, is 136×6 , or 816 gallons, the capacity of the tank. The guess of 800 gallons shows that the answer is reasonable.

One important point to note is that all the quantities in the statement showing the relations were gallons; that is:

$\frac{1}{2}$ tank (in gal.) less 136 gal. equals $\frac{1}{3}$ tank (in gal.)

When we shortened the sentences into words, these words became symbols which represented the whole sentences. When we substituted signs such as $-$ and $=$ for the words *are removed*, or *are taken away*, or *the result is*, these signs became symbols which to us had a full meaning. We used symbols because, when the statement of the whole problem is reduced to a few letters or signs which show the operations to be performed, the plan becomes clearer. We know what to do. Solving the problem directly from the first statement was difficult. By rewriting it, without any great mental exertion we made an easy problem of it.

The method illustrated above may be called the *symbolic method* of simplifying the problem by substituting letters or figures for words representing quantities, and signs or symbols like $-$ and $=$ for words showing the relations existing between the quantities and the operations to be performed upon the quantities.

You should note that the signs or symbols $+$, $-$, $\times$, and the rest of the signs used in arithmetic are symbols for word directions telling you what to do with the numbers or quantities.¹ Also note that they may be made to represent many operations described in different words. Thus, the sign or symbol $-$ may stand for *minus*, *less*, *take away*, *subtract*, *diminish*, *decrease by*,

¹ The words *sign* and *symbol* are repeated together in order to make the meaning of the rather unfamiliar word *symbol* clear to the learner.

remove, go down, to the left, backward, and other words having similar meanings.

In like manner the sign or symbol $+$ may stand for *plus, add, and, increase, increased by, go ahead, to the right, go up.*

And the symbol $=$ may stand for the following meanings: *equals, is the same as, is, what is left is, the result is, balances, gives, leaving, makes, and so on.*

The symbol $\div$ or $/$ may stand for *per, for each, divide by.*

You should make a list of the various meanings of these operating symbols from the problems that you solve, as you will find that when you recognize these phrases in problem statements they will become hints to solutions. That is, recognition of these words that can be replaced by symbols greatly facilitates the translation of verbal into symbolic statements.

We begin to see how mathematics is applied to the solution of problems.

1. Every problem may be considered as a statement or a group of statements, or as a description of some situation.
2. We translate such verbal statements into symbolic statements, which show at once the relations between the quantities described, and also the operations which must be performed.
3. We then manipulate these symbols for the purpose of obtaining a numerical solution.

Discussion. The estimate of the answer should be insisted upon. Its value is great in the beginning of the term for the following reasons:

1. It obliges the pupil to read the problem attentively at least twice. This increases the chances of solution.
2. It promotes habits of suspended judgment by showing him that his guess is of little value. It weakens his complacent attitude.
3. It livens the class-work. The various guesses, all tabulated on the blackboard, arouse a great deal of interest.
4. It is excellent practice in perceiving relations between magnitudes or between parts of figures.
5. It develops the ability to grasp a situation as a whole.

Rewriting and restating a problem, dividing it into separate statements, is a very valuable classroom device, which will be used throughout this book. Restating a problem, rearranging the words, taking out the unnecessary words, introducing new and more significant words, all provide a good form of exercise in language and thought training. These processes promote valuable habits of precise thinking and word analysis. It stands to reason that before expressing relations by means of symbols, it is necessary to express them in English.

The setting of a problem, that which makes it interesting by showing its relation to life, becomes an ob-

struction to solution as soon as it has served its motivating purpose. The unnecessary details distract attention. That which is material is obscured by that which is irrelevant. The teacher cannot pay too much attention to this apparently small point. The pupils should be trained to discover the important points and to eliminate the non-essential.

This is one of the reasons why textbooks in "practical" mathematics produce such poor results. The students pay more attention to the setting than they do to the method of solution. Diagrams and sketches are also bewildering to the beginner until he is trained to isolate the particular part that is to be studied. For this reason elaborate drawings in mathematical textbooks should always be accompanied by a skeleton design which clearly brings out the essential points, lines, or angles. Training in abstraction should not be confined to words and sentences only.

Giving various meanings to symbols will be found to be of great assistance in problem-solving. These meanings should be copied in large characters on a large sheet of paper which can be left in full sight of the class. Each pupil should be instructed to make a copy of it on a loose sheet of paper which should also contain a list of the important formulas and the skeleton of solution described later.

Analysis of a problem from another point of view

A problem contains two essential parts: a question,

and the answer to the question. This answer is commonly called the *statement of the problem, the description of the conditions*, etc. In the final analysis it is only the distorted answer. An example will justify this statement:

Question. What is the capacity of this tank in gallons?

Answer. 840 gallons.

Question. Same as above.

Answer. One half of the tank contains 420 gallons.

Question. Same as above.

Answer. One sixth of the tank contains 140 gallons.

Question. Same as above.

Answer. One half of the tank less one third of the tank is 140 gallons.

Question. Same as above.

Answer. If from one half of the tank we take away 140 gallons, what is left is one third of the tank.

And the answers may become more and more complicated and distorted, and still remain answers. That is, we reach the same conclusion as before: The problem contains everything necessary for the answer.

By a mathematical operation, then, we mean any process through which the answer is turned, rearranged, simplified, etc., until it becomes direct. The easiest way of obtaining this result is to put the answer in symbolic form, because symbols are easier to

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manipulate than thoughts. They can be written quickly and are amenable to definite laws.

EXERCISES

The following exercises provide practice in translation of word-statements into symbolic statements. They also aim to motivate the symbolic manipulation drills given later. For this reason the students should be told that solutions need not be obtained. They should also be directed to rewrite the statements if necessary and to use the list of symbol meanings given in this chapter.

1. If you add 6 to the age of John the result is 19.

Form of work:

The symbol for *add* is $+$, for *the result is* is $=$, so we write,
 $J + 6 = 19$ Ans.

2. 19 exceeds Sam's age by 4 years.
3. Three times a certain number is equal to the number increased by 20.
4. The area of a rectangle is the product of the length by the width.
5. The area of a triangle is one half of the base times the altitude.
6. The quotient of two numbers is 7. The divisor is 4.
7. Three fourths of the number of pupils in a class is 15.
8. The distance traveled is equal to the product of the time by the rate.
9. The traveling rate is equal to the distance divided by the time.
10. The time spent in traveling is the quotient of the distance divided by the rate.
11. Twice my age diminished by 9 is equal to my age increased by 14.
12. If 6 is taken from 3 times a boy's age, the result is the same as when the boy's age is taken from 14.
13. If a certain number is added to $\frac{1}{2}$ of itself, the sum is 20.
14. One half of a number added to one third of it makes 10.
15. Double my age and add one half of it; the result is 35.
16. A wagon loaded with 12 sacks of flour weighs 2044 pounds. The weight of the wagon is 800 pounds.
17. A book has 355 pages. The total thickness of the book is $1\frac{1}{2}$ inches. The covers are each $\frac{1}{8}$ of an inch thick.
18. The net cost of 5 packages, after a 50 cent reduction in price has been allowed on each, is \$6.40.

19. The price of sugar is raised 10%. The new price is \$7.70.
20. One twelfth of the yearly income of a family is spent for clothing, one third for food, and one sixth for rent. This leaves \$840 for all other expenses.
21. To find the weight of 1 bolt, we put 20 bolts into the left pan of a balance and a 2-pound weight into the right. The right pan goes down, so we add 2 ounces to the left. The pans now balance. (One pound equals 16 ounces.)
22. A telegraph pole in a swamp stands one quarter of its length in the ground, one third in water, and 20 feet above the surface of the water.
23. A professor of mathematics visiting a class asked the number of pupils enrolled. One of the boys said, "Our number, and the number again, and its half, and its fourth, and one more, makes 100."
24. A sidewalk is 32 feet long and 5 feet wide. A boy sweeps it, cleaning 5 square feet at each stroke.
25. From a stack of writing paper containing 140 sheets I gave 3 sheets to each pupil and had left 74 sheets.
26. A stationer sold a quantity of fountain pens at \$2.00 apiece. He had bought them for \$46.50. His profit was \$31.50.
27. A man borrowed as much money as he had, and then spent one third of the whole sum. He had left \$4.00.
28. A tree 45 feet high was broken off so that the part left standing was twice the length of the part broken off.
29. A launch that can make 12 miles an hour in still water travels upstream for 3 hours. The total distance covered is 27 miles.

PROBLEMS

The following problems contain two or more statements; that is, they must be translated into two or more equations. *Do not try to express all the unknowns in terms of one.*

1. A box and its contents weigh 56 pounds. The contents weigh 3 times as much as the box.
 Statement 1. $B + C = 56$
 Statement 2. $C = 3B$
2. The total number of children in a school is 452. There are 40 more boys than girls.
3. At a school election two opposing candidates receive together 548 votes. One candidate received 100 more votes than the other.

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4. Jim and Sam have together \$8.00; Jim and Fred, \$10.00; Sam and Fred, \$12.00.
5. A farmer sold a sheep, a cow, and a horse for \$216. He sold the cow for seven times as much as the sheep, and the horse for four times as much as the cow.
6. A sum of money was divided among Jim, Sam, and Fred in such a way that Jim received three times as much as Sam, and Sam twice as much as Fred. Jim received \$6.00 more than Fred.
7. Three boys did a certain job together, for which they were paid \$24.00. B did one third of what A did. A and B together did twice as much as C.
8. The perimeter of a rectangle is 96 feet. The altitude is one half the base.
9. A and B together own a business worth \$16,500. B's share is twice that of A.
10. A city block is $81\frac{1}{2}$ feet longer than it is wide, and the length and width together measure $551\frac{1}{2}$ feet.
11. John had four times as many marbles as Robert. After losing 14, he had 3 times as many.
12. One faucet runs water into a bath tub twice as fast as does another. Together they take 4 minutes to fill the tub, which holds 100 gallons.

CHAPTER IV

INTRODUCTION TO THE EQUATION AND ITS LAW

The balance method

THE *balance method*, or the use of the balance in introducing the concept of the equation and its law, is excellent. It is fundamentally sound, because it compares the equation to a familiar object, and associates it with a fact that every child knows — that a change on one side of the balance must be accompanied by a corresponding change on the other, if equilibrium and equality are maintained. The method of presentation is well described in several modern textbooks. One reason for describing it in detail in this chapter is that the addition of the simplification-of-statement process and of the short statement of the Equation Law to the usual balance method produce somewhat better results than the customary procedure.

Besides the balance method the reader will find four other methods of establishing the equation concept. While these are inferior to the balance method, they are valuable in that they confirm and generalize the initially established notion that the symbol $=$ stands for equality between the two sides of the equation. That is, the pupil learns that the symbol $=$ means equality of such quantities as length, cost, volume, etc., as well as equality of weights.

It is believed that the Equation Law is more com-

prehensible and serviceable when reduced to a short sentence than when expressed in the form of four axioms. Besides this, the author challenges the assumption that those axioms are self-evident to children of normal mentality. Many children no more know that things equal to the same thing are equal to each other, until they see it demonstrated and think it over, than they know that yellow and blue make green. In the short universal form here given the Law of the Equation becomes authority for all symbolic transformation processes from the simplest equation to the most difficult differentiation or integration.

Classroom presentation. The following technique is used in explaining symbolic transformation in the early part of the subject:

Example. $3x + 7 = 22$; find the value of x .

Our purpose is to find the value of x . By this we mean that x shall be made to stand alone on one side of the equation. The other letters or numbers prevent this. Let us get rid of them. To get rid of them, we have to follow one rule, the *Equation Law*, which states that *whatever we do to one side of the equation we must also do to the other*. Now this 7 is in our way. We do not want it. Let us remove it. How? By taking 7 away from it. But the Law says that we must take away the same amount from the other side, lest the balance be destroyed. (Recall of fundamental concrete experience.)

We want to get rid of this 3 which multiplies the x . Let us remove it. How can we do this? By dividing it by 3. But the Law states . . . and so on.

We use four methods for removing numbers that are in our way, according to the part that they play in the equation. How do I know which to use? I follow one rule: perform that operation which is opposite to the operation indicated by the sign connecting the undesired number. If this sign is $-$, I perform the opposite operation and add. If it is $+$, or not expressed, the sign $+$ being understood, I subtract. If it is $\times$, I divide, and if it is $\div$, I multiply.

That is, we have one law to obey, the Law of Equation; and one rule to follow: to remove a number, perform the operation opposite to the operation indicated in the expression.

Discussion. This procedure should be adhered to until the children understand the routine perfectly. It may be found advisable to place the sign $+$ before the first term and the sign $\times$ between factors until the operation is thoroughly understood. Otherwise, at the end of the course the teacher may still find some children who do not know how to proceed. *Constant repetition of essential processes of the above type is imperative.*

Example 1. Fig. 1 shows a balance. In one pan there are 5 large bolts. In the other pan there is a weight of 20 pounds. How much does each bolt weigh? Answer, 4 pounds.

How did you obtain this result? You divided 20 by 5. Why? What you really did was to divide the weight of 5 bolts by 5 to obtain the weight of one bolt;

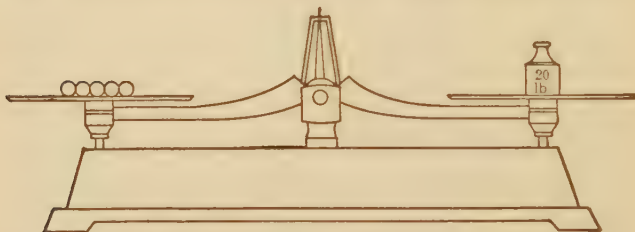


FIGURE 1

and to divide the number of pounds in the other scale pan by 5 also. One bolt in the one pan now balances a weight of 4 pounds in the other pan. Having performed the same operation on the contents of each pan, you have not destroyed their balance.

We may simplify our problem into one sentence which fully describes the situation:

The weight of five bolts is twenty pounds.

Or, in symbols,

$$5b = 20,$$

provided that we agree to have the symbol = mean *balances*, or *have the same weight as*, or *equals*; to have the symbols 5 and 20 stand for the words *five* and *twenty*, and the letter *b* stand for the number of pounds that each bolt weighs. Such a symbolic translation of a statement in which two expressions have the same

value (in terms of some definite unit of measurement) is called an *equation*. It shows by means of symbols or signs the meaning of a word-statement. That is, it gives a clear, concise, usable expression of the situation. It is useful because it shows at a glance what arithmetic or algebraic operations should be performed in order to obtain the result desired. In our problem the desired result is the weight of one bolt, or, symbolically, the value of b .

$$5b = 20$$

We want 1 b . The factor 5 is not wanted. To remove it, we divide it by 5; that is, by the same number. But the moment we divide by 5 on one side, the balance is destroyed; there is only one bolt left in the one pan, while the twenty pounds are still in the other pan. We must divide the 20 in the other pan by 5 also, or divide the other side of the equation by 5. The balance or equality is then reestablished.

$$\frac{5b}{5} = \frac{20}{5} \quad \text{or} \quad b = 4$$

which we translate back into words; one bolt weighs four pounds.

This suggests a thought: Dividing each side of an equation by the same number does not destroy the equality. As this always holds true, we may consider it a rule.

This rule, however, is only a partial application of

the following general rule which governs all the operations performed on equations:

Any operation performed upon one side of the equation must also be performed upon the other side

Or, Whatever is done to one side of the equation must also be done to the other side. Or, Like changes must be made on each side. The balance must be preserved.

This is the fundamental rule or Law of Equations. All the other rules or laws are only partial applications to special cases. Thus we may say: If we add some number to one side of the equation, we must add the same number to the other side. If we multiply or divide one side of the equation by some number, we must multiply or divide the other side by the same number. If we subtract some amount from one side of the equation, we must subtract the same amount from the other side, and so on.

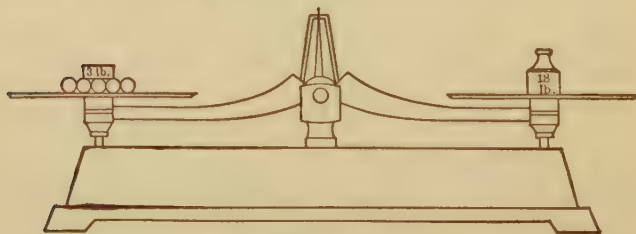


FIGURE 2

Example 2. Fig. 2 shows some bolts of another kind being weighed. In one pan we have 5 bolts and a

3-pound weight; in the other pan we have 18 pounds. How much does each bolt weigh?

We first simplify the statement:

The weight of 5 bolts and 3 pounds balances 18 pounds.

Replacing by symbols,

$$5b + 3 = 18$$

The 3 on the left side of the equation is in the way. The Law of Equations states that we can take away this 3, provided we also take away 3 from the other side. We choose then to take 3 away from each side; that is, to subtract 3 from each side. We have left,

$$5b = 15$$

To get rid of the 5 we apply the law again and divide each side by 5. This gives us

$$\frac{5b}{5} = \frac{15}{5} \quad \text{or,} \quad b = 3$$

which we can translate into words: One bolt weighs 3 pounds.

In order to make sure that we have the right result, we examine the statement of our problem. Each bolt weighs 3 pounds. Five bolts weigh 15 pounds, 15 and 3 balance 18 pounds. This is our check.

The same method may be applied to other problems. Thus:

Example 3. A man bought 6 hinges. He gave the

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storekeeper 50 cents and received 2 cents in change. What was the cost of one hinge? Rewrite and simplify:

50 cents have the same value as 6 hinges and 2 cents

In symbols,

$$50 = 6h + 2$$

or,

$$50 = 6h + 2$$

Taking 2 away from each side,

$$\begin{array}{r} -2 \quad -2 \\ \hline 48 = 6h \end{array}$$

Dividing each side by 6,

$$8 = h$$

Translating back into words: Eight cents is the cost of one hinge.

Check. On re-reading the problem, we find that this value satisfies the given conditions. Six hinges, at eight cents each, and two cents, amount to fifty cents.

Example 4. Thirteen nails are placed in a single line with the ends touching. The line lacks ten inches of being one yard long. How long is each nail? (One yard = 36 inches.)

Rewrite and simplify:

The length in inches of 13 nails and 10 in. make 36 in.
In symbols,

$$13 n + 10 = 36$$

or,

$$13 n + 10 = 36$$

Taking away 10 from each side and then dividing each side by 13 are evidently the next steps. You can see now that putting the problem into a symbolic form automatically gives you the plan of solution. The signs tell you what to do.

Example 5. A bucket holds 21 pints of water. We fill a bottle 6 times from it and have 3 pints left in the bucket. How many pints does the bottle hold?

Simplifying:

The number of pints in 6 bottles and 3 pints is the number of pints in the bucket,

$$6 b + 3 = 21$$

or,

$$6 b + 3 = 21$$

The rest of the solution is as above. We subtract 3 from each side, then divide each side by 6, the object being to get b alone on one side of the equation.

Example 6. Fig. 3 shows 4 books (of equal weight) and 3 pounds in one pan of the scales, and 3 books of the same weight and 5 pounds in the other pan. How much does each book weigh?

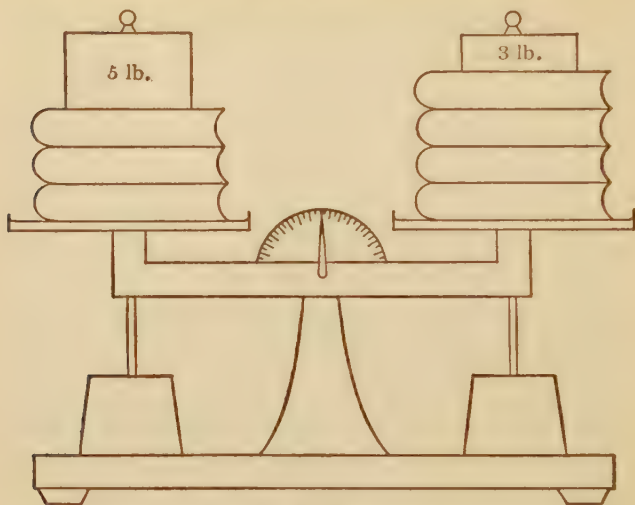


FIGURE 3

Simplifying:

4 books and 3 pounds balance 3 books and 5 pounds

In symbols,

$$4b + 3 = 3b + 5$$

or,

$$4b + 3 = 3b + 5$$

Taking $3b$ away from each side, gives us

$$b + 3 = 5$$

Then taking 3 from each side leaves us

$$b = 2$$

Translating into words: One book weighs 2 pounds.

Example 7. Fig. 4 shows a little emergency weighing situation. The package p is too heavy to weigh on the scales directly. A beam b is used, one end of which rests on the block w and the other on the scales.

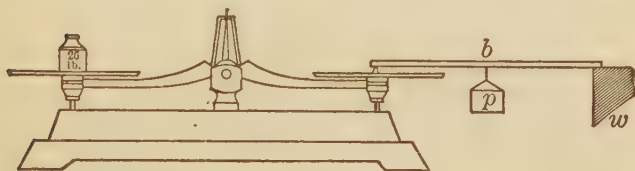


FIGURE 4

The package is hung in the middle, so the scales show exactly one half of the weight of the package after the pressure due to the beam has been found to be 6 pounds. A weight of 26 pounds secures the balance. What is the weight of the package? (The weight of the beam is to be neglected.)

Simplifying the statement:

One half weight package and 6 lb. bal. 26 lb.

or,

$$\frac{1}{2} p + 6 = 26$$

or,

$$\frac{1}{2} p + 6 = 26$$

Subtract 6 from each side, :

$$\frac{1}{2} p = 20$$

One half of the package weighs 20 pounds, how much

does the whole package weigh? We all know the answer, but let us try to find it from the equation; because, while we can solve an easy problem mentally without knowing exactly how we do it, when we have a more complicated problem we must have a method which we know will yield results.

We must obtain $p = 40$. The only way in which we can obtain this is by multiplying each side by 2. Then:

$$2 \times \frac{1}{2} p = 20 \times 2 \quad \text{or} \quad p = 40$$

Discussion. It becomes advisable now to assign drill problems. Good results can be obtained from these if the equations derived from the problems at the end of Chapter III are used, as the usefulness of the drill is thus emphasized. The pupils will form the all-important notion that an equation is only a symbolic form of a statement. It is suggested that at least a week be given to individual assistance during class periods, instead of the usual recitation procedure. The teacher will thus be enabled to judge the capacities of the various individuals, and also the results of the presentation.

Conditions essential to effective drill

Additional drill exercises are not given here, as practically every textbook has this aspect of the subject well organized. If anything, it is too well organized, for too many exercises are given, some of which

require an amount of time out of proportion to their usefulness. The important points here are the attitude toward the drill, the directions given, and the extent to which they are followed, and not the contents. If the child does not perceive the necessity for the drill, the facility acquired under compulsion is of doubtful value and easily lost. By significant drill, then, we mean a process during which the objective is constantly kept in mind and certain steps are mentally repeated. It is the lack of this mental recall that makes the average drill of so little value. The description of a specific case will illustrate:

Illustration of defective training

Two high-school students were found to be unable to perceive how

$$\frac{7}{2} \times \frac{3}{2}$$

could be equal to $5\frac{1}{4}$. Searching questioning brought out the following facts:

1. They had worked very many exercises in multiplication of fractions in their earlier school work. These exercises were given in groups of twenty or more, speed being the main aim.
2. They had solved very many reduction-to-improper-fraction examples, also in groups, also for speed.
3. They could quote correctly the rules applied.

But the three had been "learned" as separate units, and hence did not function together.

The process involved was thoroughly explained to both students, beginning with the conception of the fraction, and a number of drill exercises were assigned them. Directions for mentally repeating the rule and the reason for the drill before starting each example were given.

One of the students followed directions and later applied the same procedure to all drill work. Although rather dull, he acquired mastery over mathematical processes. The other student was not persuaded that he needed to change his method of learning. He handed in the examples, claiming that he had followed directions. Similar deficiencies in his mathematical training were found later and similar attempts were made to correct them, with similar results. He did the work by imitation and refused to think or to associate the process with the rule. Only his failure to obtain a passing grade brought him to reason. He then acknowledged his disregard of directions, and eventually became an enthusiastic and successful student of mathematics, which subject he had previously "hated."

An experiment in teaching mathematics

The author does not believe in assigning drill in symbolic manipulation before showing how equations are derived from word-problems. He has taught one

class the process of translation of word-problems into equations before teaching symbolic manipulation systematically. It was quite interesting to observe how some students indicated the multiplication of monomials and binomials without the use of exponents and parentheses, and solved equations by improvised common-sense methods.

Eventually some of the boys complained and asked for systematic instruction in common processes. Some were even found to have learned the processes and drilled on them, without being taught and in addition to the regular assignments. The class was evidently ripe for intensive drill work. A few words of instruction and directions as to how to obtain the best results from the drill were given, and the class started on the go-as-you-please plan. How they did cover the ground! It was not uncommon for the children to bring in several pages covered with examples, or even all the examples in a chapter. Drill under such conditions is a pleasure, as the child feels the satisfaction of overcoming a difficulty voluntarily, and is conscious of his growing power. The usual meaningless and obligatory drill is drudgery.

Kinds of drill

Besides the well-known drill examples in simple and simultaneous equations, factoring, etc., given in modern textbooks, valuable drills can be assigned in the following processes:

1. *Drill in analysis* of the situations described in the problems; in rejection of non-essentials; in reduction to mathematical terms.

This is especially valuable toward the end of the course, when practical problems are studied. A few days devoted to the examination of a large number of such problems without the actual solution will prove to be of inestimable value for rounding off the course in the case of pupils who will not take more mathematics, as well as for those who intend to pursue the study in higher branches. The latter will be ready to begin the next term's work with the right attitude. It is sometimes feasible to get assistance from the English department in this kind of drill.

2. *Translation of statements into symbols*, without complete solution.

An example of this is given in the problems at the end of Chapter III. This should be given very early in the term. It has been found to be of high value in motivating the drill in simple and simultaneous equations.

3. *Drill in finding implied statements*, and in separating the problem into statements describing or relating the quantities.

This will be described in Chapter VII.

CHAPTER V

THE SOLUTION OF PROBLEMS — SETTING VERSUS METHOD

Simultaneous equations in the traditional arrangement and in modern procedure

TRADITION places the study of the simultaneous equation in the middle of the one-year course, the various methods of solution being grouped together. This arrangement was logically correct when symbolic solution was taught before any problems were presented. But this sequence causes considerable trouble in the earlier parts of the subject now that problems are presented from the first. Restriction to problems which can be expressed at once in one equation, combined with the unwise addition of difficult problems, has made teaching of these earlier parts arduous, dull, and discouraging. What teacher, when asked by the class to solve some complex problem by the one-equation method, has not wished for the simpler and more effective many-equation method! How often the time of a class has been wasted by fruitless efforts to "write down" an equation in terms of one unknown! Why is the comparison of ratios taught in the seventh grade considered too hard for high-school freshmen? Why is substituting on paper considered more difficult than substituting mentally, as when the student is told to "express" the other unknowns in terms of one?

An experiment was recently conducted to test the soundness of the traditional sequence. Two classes of about the same average ability with the same teacher were taught, one in the usual sequence, while the other was introduced very early to simultaneous equations. It was soon found that the many-equation method was greatly superior in that it allowed the use of a larger number of illustrative problems. It was found especially valuable in justifying the use of parentheses and in illustrating the change of signs in negative quantities inside of parentheses. It facilitated the work so much that by the end of the term one class led the other by so wide a margin in amount of ground covered and in thoroughness that even superior ability could not be claimed as the determining factor in the case.

Argument for the early introduction of simultaneous equations

A careful examination of most of the problems classified as "simple-equation" problems will convince the reader that they are only simultaneous-equation problems in disguise. They contain two or more separate statements. Solving such problems by using one unknown, and expressing the other unknowns at once in terms of that unknown in one equation, is a far more difficult process to learn than the substitution method of simultaneous equations, when one equation is written for each statement.

Moreover, the latter method is the one used in

formula work. Practical users of algebra naturally fall into the habit of writing down all of the equations and substituting into the most promising one. Why, then, teach a student a method which will not be used subsequently, not even in his later study, when he has fully learned all the other methods of solution of simultaneous equations? The most that can be said for it is that it is good mental training, the excuse usually given when no other excuse is available.

Furthermore, if the student does not retain anything from his study of mathematics excepting a thorough understanding of the methods used in formula work, substitution, and evaluation, his time will not have been wholly wasted. An early introduction means, then, in the case of most students, a large amount of practice in these essential processes.

Finally, the simultaneous-equation device is an integral part of the functional method of solving problems. The traditional method of expressing all the unknowns in terms of one has probably been the means of preventing the development of this method.

Besides mental training, the only strong argument in favor of the traditional sequence is that negative quantities must be studied before simultaneous equations. But even this is not a good argument, as the concept of the negative quantity is best acquired by seeing it used. It has been found that the formal study of negative numbers is easier after the idea has been informally acquired.

The pupil's attitude toward academic problems

Having justified the use of simultaneous equations near the beginning of the subject, we pass to another troublesome point, the attitude of children toward problems.

At first the class will accept any kind of problem; then questions begin to arise. Is this what algebra is for? — to solve problems about clocks, railroad trains, ages of fathers and sons, or about capitalists who forget how much money they have invested and want to calculate it from the interest received? Truly the "life problems" given by most books are wonderful! If the teacher incautiously promises that presently the problems will become more practical, he gets into deeper trouble, because some imp is sure to examine the back of the book and find that the problems there are of the same type, a little more complex, to be sure, but just as unpractical.

Practical problems are unsuitable for beginners

If in desperation the teacher turns to practical textbooks, he finds that he has only jumped from the frying-pan into the fire. The problems there are too difficult for his class to solve. They necessitate a process of translation into mathematical language which can be learned only through long practice and special training. If in spite of this he tries to teach their solution by means of formulas, he will discover that at the end of the year the children can only solve problems of

a very commonplace type. They have learned the problems, not a general method through which problems are solved.

Still, we can hardly blame children if they do not take much interest in problems of this sort: "One number is five times another, and their sum is thirty. Find the numbers." The teacher does not obtain much help from textbooks when he tries to persuade children that such problems are necessary steps to more difficult problems, that they lead to real life problems, and that the method of solution is of far more importance than the wording of the problem. Very few textbooks attempt to show the method underlying the solution of word-problems.

Changing the setting of a problem

It will be shown later that, once certain fundamental relations between quantities are well understood by the teacher, he can easily give almost any problem in any book a convincingly practical setting. A few illustrations of such transformations will be given occasionally in this book. These will usually show that the statement of one problem may be totally different from that of another, yet both problems be solved by identical methods, using identical coefficients. At the beginning of the term the following series usually convince pupils that the method of solution is of more importance than the verbal setting.

Classroom presentation of various settings of one

problem. (Problems to be written on the black-board.)

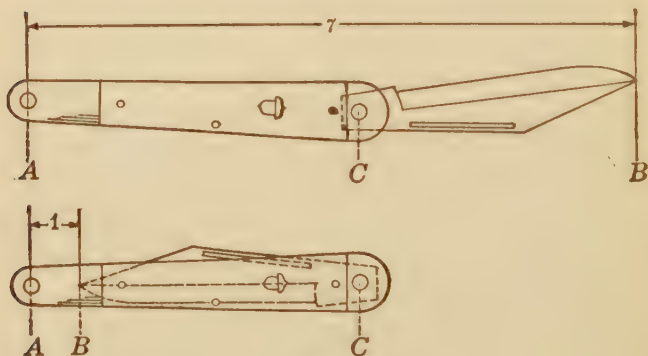


FIGURE 5

Example 1. Fig. 5 shows a pocketknife open and closed. When it is open, the total distance between pin A and point B is 7 inches. When closed, the distance from point B to pin A is 1 inch. How long is the blade, and how far is it between pins A and C ?

This problem does not seem very useful or practical, but when we compare it with the next problem, we find that the method of solution is the same. The same problem is often used in mechanics; for instance, in the design of connecting rods and cranks of engines.

Example 2. Find the lengths of the two links shown in Fig. 6. When the links are open (like an open pocketknife) the total length AB is 7 inches. When closed, the distance from A to B is 1 inch.

Example 3. In a machine two forces act differently at different times, sometimes acting in the same direction and sometimes oppositely. When acting together, they exert a total force of 7 tons; when opposed their combined effect is 1 ton. Find the forces.

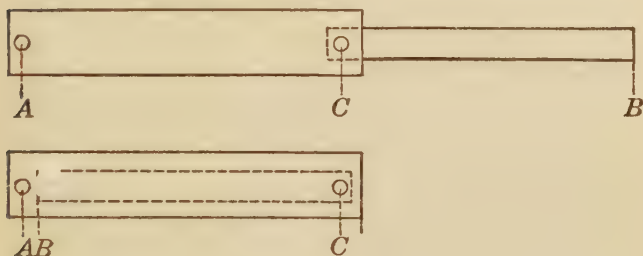


FIGURE 6

We may write another problem as follows, using the same figures and obtaining the same numerical result.

Example 4. The sum of the ages of two children is 7 years; the difference between their ages is 1 year. What are their ages?

It is clear that this problem can be stated in terms of any quantity, such as cost, speed, volume, etc. But finally we can write it thus:

Example 5. The sum of two numbers is 7; their difference is 1. What are the numbers?

This last problem is easily solved mentally. One of the numbers is 4 and the other 3. On looking over the preceding statements, we find that these numbers also satisfy the conditions described. Which of the pro-

blems was easiest to solve? Why was it easiest? It was the easiest because it was stripped of attention-distracting details. In the last, how did you obtain the answer? You must have performed some mental computation or else guessed the answer. If you guessed, can the guessing method be applied to more complicated problems of the same type? Can it be applied, for instance, to:

The cost of a motor-cycle with a side-car is \$167, and the motor-cycle costs \$53 more than the side-car. What is the cost of each?

It is clear that this problem can be written thus:

The sum of two numbers is 167 and their difference is 53. What are the numbers?

In this form it is far easier of solution. We learn, then, that problems containing numbers only are the easiest to solve. We therefore conclude that, if we wish to learn to solve problems, the easiest way is first to learn the method of solution when numbers only are used, because we can direct all of our attention to the method, and no effort need be wasted on an irrelevant setting.

Another illustration will now be given, of how problems stated in different words and describing totally different situations are yet solved by exactly similar steps and yield the same numerical results.

Example 1. A machinist wanted to find the weight of some large bolts and nuts. Not having any scales, he rigged up a balance as shown in Fig. 7a. Not hav-



FIGURE 7a

in 2 weights, he used a bag of nails which was known to weigh 3 pounds. After several trials he found that 5 bolts balanced 4 nuts and 3 pounds. In another trial he found that 3 bolts and the 3-pound weight balanced 3 nuts (Fig. 7b). Show how from these measurements he found the weights of the bolts and nuts.

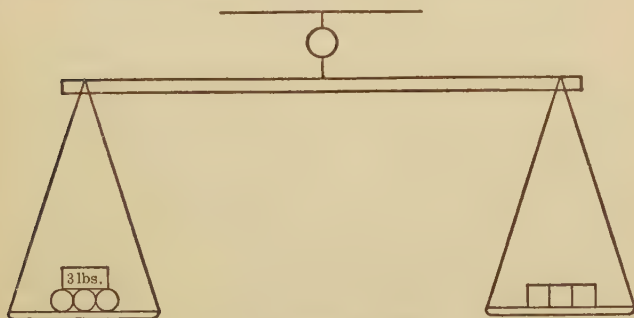


FIGURE 7b

Solution: We find that we can make two statements about the quantities described (the weights of the bolts and nuts):

1. Five bolts balance four nuts and three pounds.

Or, in symbols:

$$5 \quad b \quad = \quad 4 \quad n \quad + \quad 3 \quad (1)$$

2. Three bolts and three pounds balance three nuts.

Or,

$$3 \quad b \quad + \quad 3 \quad = \quad 3 \quad n \quad (2)$$

Our purpose now is to combine these two equations in such a way that one of the unknowns is eliminated (got rid of).

Dividing each side of (2) by 3, we have

$$b + 1 = n$$

Multiplying each side of this equation by 4,

$$4 \, b + 4 = 4 \, n$$

Now we have a value of $4 \, n$. We can substitute this value of $4 \, n$ in (1); that is, we can write $4 \, b + 4$ instead of $4 \, n$ in (1), which then becomes

$$5 \, b = 4 \, b + 4 + 3$$

or,

$$5 \, b = 4 \, b + 7$$

Subtracting $4 \, b$ from each side of this equation we get

$$b = 7$$

or, 1 bolt weighs 7 pounds. Using this value in (2), we have (since 3 bolts weigh 21 pounds),

$$21 + 3 = 3n$$

or, 1 nut weighs 8 pounds.

Check: Substituting these values in (1), we have

$$5 \times 7 = 4 \times 8 + 3$$

This balances, showing that our solution is correct, for the equation is satisfied.

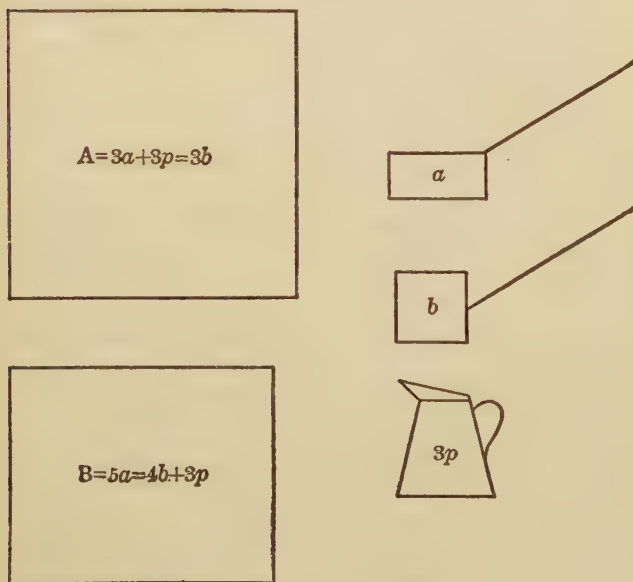


FIGURE 8

Example 2. A man had some copper tanks and dip-pers, as shown in Fig. 8. He wished to find how many

pints they would hold. The only accurately gauged implement that he had was a measure holding three pints. Using the dippers and measure, he found that tank A holds exactly the contents of three dippers, a , and three pints. In another trial he found that tank A could also be exactly filled with the contents of three dippers, b . In a similar way tank B could be filled either with 5 dippers, a , or 4 dippers, b , and 3 pints. What are the capacities of the tanks and dippers?

We find that we can make two sets of statements:

1. Three dippers, a , and three pints fill tank A.

Three dippers, b , fill tank A.

Since three dippers, a , and three pints or three dippers, b , can fill the tank A, the total quantities must be the same; that is:

Three dippers, a , and three pints equal three dippers, b .

In symbols:

$$3a + 3 = 3b \quad (1)$$

2. Five dippers, a , fill the tank B.

Four dippers, b , and three pints fill tank B.

As before, the two quantities are equal, since they each fill tank B.

Five dippers, a , equal four dippers, b , and three pints.

In symbols:

$$5a = 4b + 3 \quad (2)$$

Comparing (1) and (2) with the equations of the pre-

ceding problem, we find that they are exactly alike, with the exception of the letters used. We can see that the solution can be obtained in an exactly similar way, and the same numbers will appear. That is, $a = 7$, and $b = 8$.

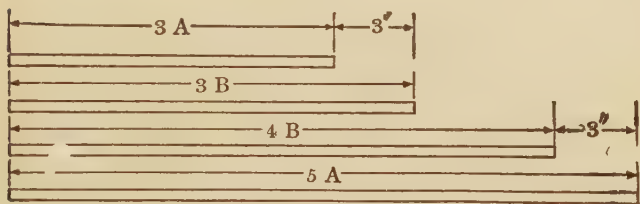


FIGURE 9

Example 3. A furniture man had some brass rods of two different lengths, a and b . The only measuring implement at his disposal was the end edge of an envelope which he knew to be exactly three inches in width. In order to measure the rods he proceeded as follows: He placed the rods, a , as shown in Fig. 9, with the ends touching. He placed the rods, b , beside them. After several trials he found that three rods, a , and three inches were exactly of the same length as three rods, b . In another trial he found that five rods, a , measured exactly the same length as four rods, b , with three inches added. What was the length of the rods? (Using sticks, this makes a good demonstration problem.)

Clearly this problem is solved by using the very same equations that we used in Example 2, and clearly

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we shall obtain the same numerical results. Rod a is 7 inches long; rod b 8 inches long.

Example 4. A draftsman must design some gears in such a way that 3 revolutions of the first gear and 3 teeth are equal to 3 revolutions of the second gear, and 5 revolutions of the first gear are equal to 4 revolutions of the second gear and 3 teeth. How many teeth must there be on each gear?

The conditions and numbers are obviously so much like those of the preceding problems that no comment is needed.

Example 5. Two boys went to a store to buy marbles. One found that he had paid 3 cents more for 3 glass marbles than he had paid for 3 china marbles. The other boy found that he had paid 3 cents more for 5 china marbles than he had paid for 4 glass marbles. Find the cost of each kind.

Example 6. If you add 3 to three times a certain number, you get three times a certain other number, while five times the first gives three more than four times the second number. What are the numbers?

These problems are placed together in order to show you:

- (1) That the mathematical process used in the solution of a problem can be used in a large number of problems of the same type.

- (2) That the process used is important to you; not the particular result or terms used in the description of the situation.
- (3) That number problems, not interesting, and apparently of no practical value whatever, are the easiest to solve and best show the method of solution.

The best way to learn to solve problems then is to solve them in the following order:

1. Number problems, without any distracting details; for the sake of learning the method of solution.
2. Simple and perfectly understood situations, such as boys buying marbles.
3. Everyday life situations of the kind that you understand fairly well, such as simple shop problems.
4. Industrial problems.

These are likely to contain technical terms about machinery or processes of manufacture. Unless these terms are thoroughly understood, you will have trouble in forming a clear enough idea of the situation to solve the problem. After you have solved it, you will find that you have spent much more effort on the industrial details than on the mathematics. Learn mathematics first, then learn the industrial applications of mathematics. Beginners cannot do both efficiently at the same time. Industrial problems are easily solved once the mathematical processes are so familiar that the entire attention can be focused on the industrial feature.

Discussion. Such problems have been found very effective in satisfying practical-minded boys. After seeing that the same method and the same equation will solve either a marble problem or a mechanic's problem, they will cheerfully solve the problems usually found in academic textbooks. The forbidden fruit, "practical" mathematics, will lose its fascination. But the teacher must not be deceived into believing that the fight is over. Once in a while dissatisfaction will reappear. For this reason, whenever he notices a problem that he can state in two or more different forms, using the same numbers and coefficients, he should avail himself of the opportunity to impress on the class that *the all-important thing is the method and not the setting.*

Objections to practical problems

We begin to see now why machinery and process problems used in textbooks of "practical" mathematics are not well adapted to teaching mathematical processes. In them what is important is obscured by what is pedagogically unimportant. The measurements are not easy to visualize, for they usually involve quantities and technicalities beyond the knowledge of the pupil. The whole situation does not easily lend itself to clear and accurate visualizing. Such problems require much time for realization of the situation and thus leave a comparatively short time for the solution. But in order to learn mathematics, students

should spend the larger part of their time in discovering and analyzing the relations among the quantities involved. Hence beginners need problems which bring mathematical relations among familiar quantities to the foreground, not unfamiliar settings. It will be found that whatever success has been obtained by students using "practical" books has been obtained at an expense of time and effort entirely out of proportion to the results. The same results could have been obtained in a fraction of the time if some of the theoretical aspects of the subject had first been examined and mastered.

Good drill exercises in simultaneous equations can be found in any textbook, so none are presented here. It is suggested again that the two- or more-statement problems left unfinished at the end of Chapter III be used to motivate the drill work.

Introduction to systematic methods of solving problems

In order to be of any constructive value, problem-solving should follow a definite procedure. The written work should show that the following steps have been consciously taken.

1. The guess or estimate of the answer.
2. The recognition of the quantities involved and of their units.
3. The separation of the statement into partial statements relating the quantities or completely

describing the quantities. (This will be more fully explained in Chapter VII.)

4. Translation of the partial statements into symbols.
5. Symbolic solution.
6. Check.

This is only a partial list of steps. Later a complete scheme of solution will be described.

Problem 1. A machinist and his helper earned \$264 working together on a job. If the helper received four sevenths of what the machinist received, what was his share?

Solution:

Step 1. He received about \$100.

Step 2. The quantities involved are the dollars earned by each. The unit is one dollar.

Step 3. Statements relating the quantities:

A. The machinist and helper received \$264.

B. The helper's share was four sevenths of the machinist's.

Step 4. Translation into symbolic form:

"A" of Step 3 may be restated thus: the number of dollars received by the machinist and the number received by the helper add up to \$264. Or, in symbolic form:

$$\$M + \$H = \$264$$

Or simply,

$$M + H = 264 \quad (1)$$

Statement B of Step 3 may be translated directly into symbols:

$$H = 4/7 M \quad (2)$$

Step 5. Symbolic solution:

We take the value of H obtained in (2) and place it in (1), which becomes:

$$M + 4/7 M = 264$$

$$11/7 M = 264$$

Multiplying each side of this equation by 7, then dividing each side by 11 gives us:

$$M = \frac{264 \times 7}{11} \quad \text{or} \quad M = 168$$

Which we translate back into words: the machinist gets \$168.

By subtracting the value of M from 264, we obtain 96 for the helper's share.

Step 6. *Check*: 96 is $4/7$ of 168.

It should be noted that the two statements in Step 3 are related by the fact that they are dealing with the same persons on the same job. That is, the equations are related by the fact that M and H have the same values in each.

The second equation stated that H was $4/7$ of M ; that is, that H was the same thing as $4/7 M$. Therefore $4/7 M$ can be written instead of H in the first

equation. This apparently simple thought causes trouble if left unexplained.

Problem 2. John's age exceeds Sam's by 20 years. In 15 years he will be twice as old as Sam. Find the present age of each.

Step 1. The estimate is that Sam is very young.

Step 2. The quantities involved are the ages. The unit is 1 year.

Step 3. Statements relating the quantities:

A. John's age is greater than Sam's by 20 years.

B. In 15 years John will be twice as old as Sam (is then).

Step 4. Translation into symbols:

$$\text{A. } J = S + 20 \quad (1)$$

$$\text{B. } J + 15 = 2(S + 15) \quad (2)$$

Step 5. Symbolic solution:

Substituting the value of J given by (1) in (2):

$$S + 20 + 15 = 2S + 30$$

Subtracting 30 from each side, then subtracting S from each side:

$$5 = S$$

The age of Sam is 5 years; that of John 25 years.

Step 6. *Check:* In 15 years John will be 40 and Sam will be 20; that is, John will be twice as old as Sam. This is the condition stated in the problem.

Problem 3. Three boys sold tickets for a school play. Altogether they sold 372 tickets. The first boy sold 147 more than the third, and the second 129 less than the first. How many did each sell?

Step 1. We guess that the second and third boys sold about 100 each.

Step 2. The quantities involved are the numbers of tickets sold by each. The unit is 1 ticket.

Step 3. Statements relating the quantities:

A. Three boys, A, B, and C, sold 372 tickets.

B. The first boy, A, sold 147 more than the third boy, C.

C. The second boy, B, sold 129 less than the first boy, A.

Step 4. Translation into symbols:

$$A. A + B + C = 372 \quad (1)$$

$$B. A = C + 147 \quad (2)$$

$$C. A - 129 = B \quad (3)$$

Step 5. Symbolic solution. Our aim is to form an equation containing only one unknown. Let us then solve (2) and (3) for B and C

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and substitute the values in (1). Subtracting 147 from each side of (2) gives us,

$$A - 147 = C,$$

or,

$$C = A - 147$$

(3) can be written,

$$B = A - 129$$

Substituting these values of C and B in (1), we have,

$$A + A - 129 + A - 147 = 372$$

Collecting terms:

$$3A - 276 = 372$$

Adding 276 to each side:

$$3A = 648, \text{ and } A = 216$$

From this the other values, B and C , can readily be obtained.

Step 6. Check as before.

FURTHER ILLUSTRATIVE PROBLEMS

1. One man has \$52 and saves \$2 every day. His friend has \$325 and spends \$1 every day. In how many days will they have the same amount?
2. One teacher has five times as many pupils as another. Forty-two of his pupils were transferred to the second teacher. How many pupils did each teacher have at first, if after the transfer the first teacher had twelve more pupils than did the second teacher?

3. A man borrowed as much money as he already had, then spent \$10. He again borrowed as much money as he had on hand, and again spent \$10. He then had nothing left. How much had he at first?
4. The number of canoes and launches in a lake is 505. The number of canoes is 17 less than 5 times the number of launches. How many of each are there?
5. A mechanic, a helper, and a laborer are to be paid \$368 for a contract piece of work. They agree that the mechanic shall receive four times as much as the laborer, and the helper twice as much as the laborer. What is the share of each?
6. A school board divides \$56,000 on the equipment of three schools. Park School receives \$9000 more than Lake School, and twice as much as Hill School. What amount goes to each?
7. One boy said to another, "If you give me \$1, I shall have twice as much money as you." The second boy replied, "If you give me \$1, I shall have just as much money as you." How much did each have?
8. Two men buy a truck in partnership, one paying \$75 more than the other. The total price is \$375. If each is to use the truck in proportion to his payment of the cost, how many days shall each use it every month of 25 working days?
9. A boy buying writing pads and pencils was told that 2 pads and 7 pencils would cost 31 cents, or that 3 pads and 4 pencils would cost 27 cents. What was the price of one pad? Of one pencil?
10. A man received at one time \$17 for sawing 8 cords of wood and splitting 10 cords. At another time he received \$13.50 for sawing 5 cords and splitting 12 cords at the same rate as on the previous occasion. How much was he paid per cord for each kind of work?
11. A man drove 155 miles in three days. On the second day he drove 15 miles farther than on the first, and on the third day he drove 20 miles farther than on the first. How far did he drive each day?
12. A man worked 20 days during a certain month. A part of the time he was paid \$3 per day, and the rest of the time \$4.50 per day. If his total wages came to \$75, how many days did he work at each rate?
13. A man bought an overcoat, a suit of clothes, and a pair of boots. The overcoat and boots cost \$30, the overcoat and suit cost \$55, and the suit and boots cost \$35. Find the cost of each.
14. A and B each own one half of a flock of 120 sheep. In settling

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the partnership, A takes 45 sheep, B takes 75 sheep, and gives A \$75 to make the division equal. What is the value of one sheep?

15. Out of a drum of gasoline one fifth leaked away; then ten gallons were drawn. The drum was then two thirds full. How many gallons did it hold?
16. A man leased a garage for 40 years. One half of the time the lease has yet to run is equal to one third of the time it has already run. How many years has it yet to run?

CHAPTER VI

PARENTHESES AND NEGATIVE NUMBERS

The parenthesis symbol

THE parenthesis symbol offers some difficulty to beginners. The box device described will be found to be of great help to the teacher at this juncture. If we can connect the parenthesis notion to the idea of a package, bundle, or box full of quantities that must remain bound together so long as the symbol is present, and to the idea that the number outside multiplies every number inside the parenthesis, we have solved the hardest part of our problem. The examples also help the pupil to realize that he has used parentheses unconsciously in his everyday quantitative thinking.

Classroom presentation. Example 1. It is often convenient to subtract numbers while performing some other operation. Thus, if we had 9 pieces of wood each 1 foot, 11 inches long, what would be the total length?

We may reduce to inches, then multiply the number of inches by 9, then divide by 12, obtaining 17 feet, 3 inches for the total length.

But it is shorter to reason thus:

1 foot, 11 inches is one inch less than 2 feet; 9 times 2 is 18, subtract 9 inches. We obtain 17 feet, 3 inches.

This can be performed mentally. Symbolically we would show it thus:

$$\begin{aligned} 9 \times (2' - 1'') &= 18' - 9'' \\ &= 17' + 12'' - 9'' \\ &= 17' + 3'' \end{aligned}$$

Or we may do it this way:

$$\begin{array}{r} 2' - 1'' \\ \times 9 \\ \hline 18' - 9'' \end{array}$$

The symbol () is called a "parenthesis." It means that the terms inside of it act together as a single term so far as the other terms in an equation are concerned. It is plain that $2' - 1''$ must be considered as one quantity, since it takes both of the numbers to stand for the length of one board. The 9 multiplies the $2'$ and also multiplies the $- 1''$. That is, we may say that the 9 outside multiplies every term inside.

$$9 \times (2' - 1'') = 9 \times 2' - 1'' \times 9$$

We should notice also that the feet are measured in one direction, while the inches are measured in the opposite direction. We add all the feet and subtract all the inches.

We may compare the parenthesis to a box or bundle of quantities tied together for more convenient handling. Thus, suppose that we have 5 boxes and that each box contained 4 apples and 7 pears. How many apples and how many pears would we have?

We would show the situation thus:

$$5 \times (4a + 7p) \quad \text{or} \quad 5(4a + 7p)$$

and we would know that the last expression is equal to $20a + 35p$. That is,

$$5(4a + 7p) = 20a + 35p$$

The parenthesis symbol is used whenever two or more quantities must be considered as acting as one whole in a statement. Although the student may not be aware of it, he has often used the parenthesis principle in his everyday calculations. Thus, let us suppose that some boys have sold tickets for a school entertainment at 25 cents a ticket. Sam sold 6, Fred sold 9, and Jim sold 5. How much money should they turn in?

$$\begin{aligned} 25(6 + 9 + 5) &= 25 \times 20 \\ &= 500\text{¢, or } \$5 \end{aligned}$$

We add the number of tickets and multiply the result by 25.

Example 2. What is the cost of 7 hammers at 95 cents each?

We may multiply on paper: 95×7 , which would give \$6.65, or we may say 95 cents is 5 cents less than one dollar, so the result is 35 cents less than \$7. The symbolic way of showing the last operation would be:

$$7(100 - 5) \quad \text{or} \quad 7 \times 100 - 7 \times 5$$

Example 3. Five equal-sized bags of sugar weigh together 36 pounds. Three of them are full, while 7 pounds have been removed from each of the other two. What is the weight of one full bag?

Rewriting and simplifying:

3 full bags and 2 bags from which 7 pounds are removed weigh 36 pounds

$$3b + 2(b - 7) = 36$$

Removing the parenthesis,

$$3b + 2b - 14 = 36$$

Simplifying,

$$5b - 14 = 36$$

From which we obtain

$$b = 10$$

Example 4. Seven castings in the rough and 15 pounds balance 8 finished castings and 3 pounds. A finished casting weighs 7 pounds less than a rough casting. What is the weight of a rough casting?

Simplifying, we have two statements:

Statement 1:

7 rough castings and 15 pounds balance 8 finished castings and 3 pounds

or,

$$7R + 15 = 8F + 3 \quad (1)$$

Statement 2:

The finished casting is 7 pounds lighter than the rough.

Symbolically,

$$F = -7 + R$$

or,

$$F = R - 7$$

Multiplying each side by 8,

$$8F = 8(R - 7)$$

Substituting this value of $8F$ into (1),

$$7R + 15 = 8(R - 7) + 3$$

Removing the parenthesis,¹

$$7R + 15 = 8R - 56 + 3$$

From which we get

$$R = 68$$

Example 5. A foreman gave an apprentice a rod and the following directions: "From this 16-foot board cut 5 pieces as long as this rod, 4 pieces 1 inch longer, and 6 pieces 1 inch shorter." The length of the piece of board left uncut was 33 inches. How long was the rod? Simplifying the statement:

5 pieces and 4 pieces 1 inch longer and 6 pieces 1 inch shorter and 33 inches make 192 inches.

$$5p + 4(p + 1) + 6(p - 1) + 33 = 192$$

Removing parentheses,

$$5p + 4p + 4 + 6p - 6 + 33 = 192$$

Combining,

$$15p + 37 - 6 = 192$$

$$15p + 31 = 192$$

Taking 31 away from each side, then dividing each side by 15 gives $p = 11''$.

Example 6. Two drums of equal size contain the same amount of gasoline. From one drum 37 gallons are drawn, and from the other 7 gallons are drawn. The quantity remaining in one drum is now 7 times the quantity remaining in the other. How much did each drum contain at first?

We can make this statement:

The amount left in one drum is seven times the amount left in the other:

$$D - 7 = 7(D - 37)$$

or,

$$D - 7 = 7D - 259$$

and the rest of the solution is easy.

Example 7. A man was hired for 26 days. Every day that he worked he was to receive \$3, and every day

he was idle he was to pay \$1 for his board. At the end of the month he received \$62. How many days did he work?

We can make the following statement:

The money earned, less the money paid for board, is \$62.

$$E - B = 62 \quad (1)$$

The money that he earned was \$3 multiplied by the number of days, or $3N$ dollars, if N is the number of days. The number of days that he did not work is $26 - N$. If he paid \$1 for each idle day, he paid $1 \times (26 - N)$ or simply $(26 - N)$.

Substituting these two values into our first equation, we have:

$$3N - (26 - N) = 62$$

Or, removing the parenthesis,

$$3N - 26 + N = 62$$

For $(26 - N)$ means a number " N less than 26." Hence when we subtract the 26 we have subtracted N too much, and must, therefore, *add* it.

Adding 26 to each side and combining $3N$ and N ,

$$4N = 88 \quad \text{or} \quad N = 22$$

We deduce the rule: *To remove a parenthesis preceded by the $-$ sign, change the sign before every term. The sign $+$ inside the parenthesis is changed to $-$, and the*

sign $-$ to $+$. It should be kept in mind always that the first term inside the parenthesis is supposed to have the sign $+$, unless the sign $-$ is shown.

Drill exercises in nests of parentheses are not easy to justify, as textbooks do not give practical applications. The following example (which can be modified to contain more parentheses by adding more discounts) will be of assistance to the teacher if he finds it necessary to introduce nests of parentheses.

Example. In answer to a query about the price of some machinery and tools, a shop man received a number of answers of the following type: "The catalogue price of this machine is \$214.35 less 35% and 22% off, and 5% off for cash." What is the net price of the machine? Work out a formula for solving quickly quotations of this sort.

Solution:

Instead of the price \$214.35, let us use the symbol C . Then the net price is:

$$(C - .35 C) - .22 (C - .35 C) - .05 [(C - .35 C) - .22 (C - .35 C)]$$

Since every term contains C , we can factor it out. We have then,

$$\begin{aligned} \text{Price} &= C [.65 - .22 \times .65 - .05 (.65 - .22 \times .65)] \quad (1) \\ &= C (.52 - .05 \times .52) \quad \text{or} \quad = C (.52 - .026) \\ &= 214.35 \times .494 \end{aligned}$$

Equation (1) can be factored thus,

$$P = C (1 - .05) (.65 - .22 \times .65)$$

A little study of the formula shows how it can be applied to any discounts. The .05 does not need comment, nor does .22. The .65 is obviously $1.00 - .35$.

Negative numbers

History tells us that the negative quantity was not used until the beginning of the seventeenth century. This fact of itself is sufficient evidence of the intrinsic difficulties involved in the conception of a negative number. These difficulties are responsible for a divergence of opinion as to the way in which the subject should be introduced and handled in an elementary course. The general consensus of opinion, founded on the experience of the last twenty years, seems to be that the best results are obtained by showing *how* negative numbers are used, and then showing *why* they are used. A satisfactory analysis of the fundamental principle or concept involved, through which an easy association could be made, apparently does not exist. Until one is made, we must remain satisfied with the present method.

We may, however, analyze the weak points of the present method from a common-sense point of view, and suggest slightly more effective illustrations. It is believed that the two additional illustrations described in this chapter will be found of value in overcoming some of the initial difficulties.

Analysis of traditional illustrations

Most textbooks show the use of negative numbers through the following illustrations:

1. The thermometer, when the mercury is above or below zero.
2. Financial situations, the negative sign indicating debts.
3. A time scale, the eras A.D. and B.C. being used.
4. A balloon pulling upward on some weight, or other opposing forces.
5. A distance scale or a number scale.

The first four are of doubtful value for the following reason: They try to create a mathematical notion by means of experiences or phenomena rather unfamiliar to most pupils. This may be more simply described as a process of founding deduction without perception. The inventors of these devices did not look at things from the child's point of view, and did not, apparently, realize that the objects and situations that were concrete to them were mainly abstract or even unknown to most of their pupils. Thus it was found by actual test that three fourths of the ninth-grade children in a Southern town could not remember ever having heard of a zero temperature; while one half of them did not know what a thermometer was, could not describe one, or did not understand how it worked. To illustrate an abstract idea through another abstract idea is not sound pedagogy.

The same criticism applies with equal force to the

time scale, using years A.D. and B.C. It will be found that many children do not know nor understand clearly what these letters mean, or how they are used.

The financial situation illustration is also weak because children do not see the use of the symbolic notation. To many of them debts of thousands of dollars are abstract things. The balloon illustration is weak for similar reasons. Most children's conception of a balloon is that it is a large object that floats in the air. They have no knowledge that it pulls upward. Action and reaction are not usually known to them.

The teacher is earnestly advised to make sure that explanations of fundamental notions do not themselves need explaining. The excuse that physics or history is being taught at the same time by these methods invites the retort that it is difficult enough to teach fundamentals without burdening the process with attention-distracting details. Illustrative problems should be carefully differentiated from problems showing applications.

The distance and number scales are more concrete and can be understood by all, a point in favor of the device. But it is rather weak on account of its lack of practical application. The thought induced is, "Yes, we can see that we may describe this distance from a certain point as -3 , but who ever does this?"

The introduction of the concept

After careful consideration of these objections to the

usual methods, it can be seen that problems showing the use of negative numbers should precede the usual formal introduction. Many interesting problems of this type can be found which require the methods of simultaneous equations; that is, which introduce negative numbers in a natural way. Elimination by substitution quickly teaches the use of parentheses.

Whenever tried, this method has produced good results. The children form the all-important notion that negative numbers and parentheses *are used* in problems.

After these notions are acquired, it becomes necessary to organize the knowledge and to establish the few rules and laws governing the use of negative numbers. We present this introduction in the form of a review followed by two illustrations carefully devised with due regard for the comprehension of the whole class. Going up or down stairs from a sidewalk, or the years before and after one was born, are situations which are thoroughly understood or readily visualized by every one. Such illustrations do not contain attention-distracting details; or, rather, the non-essential details disappear while the main principle remains in the foreground. It is advisable, then, to use these two devices first, to follow up with number and distance scales, and to wind up with the debts, Christian era, balloon, and thermometer devices. We cannot have too many illustrations.

Classroom presentation. Example 1. I am 14 years

old. My father built this barn 5 years after I was born. How old is the barn?

My father built this house 5 years before I was born. How old is the house?

Why do you subtract 5 from 14 in the first case, and add 5 to 14 in the second case?

What was the length of time between the building of the house and barn? Why do you add the two 5's?

My father will sell this house 5 years from now. How old will I be then?

In all the above situations I began to count time from the day that I was born. But time existed before that. So I count backward when I mention something that happened before my birth. If I put the years of my life on a line, as shown in Fig. 10, the beginning must be at "no years" or zero.



FIGURE 10

If I want to count the years before my birth on the same line, I would have to count backward, as shown in Fig. 11:



FIGURE 11

In order to distinguish the years before my birth from the years after, to show on what side of zero they are, we will put the signs + and - before the numbers. Thus the present is the year + 14, the house will be

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sold in the year $+ 19$, the house was built in the year $- 5$, the barn was built in the year $+ 5$. It can readily be seen that time and space will be saved by the use of these symbols. The sign $+$ would stand for "after I was born," and the sign $-$ would stand for "before I was born."

The operations in the first part of the example would be shown thus:

$$\begin{array}{rcl} + 14 & \text{My age} & \\ + 5 & \text{When the barn was built} & \\ \hline + 9 & \text{This must be the age of the barn} & \end{array}$$

We know that we must subtract $+ 5$ in order to obtain $+ 9$. Symbolically this operation would be shown thus:

$$\begin{array}{rcl} + 14 & & \\ - (+ 5) & \text{(subtract } + 5) & \\ \hline + 9 & \text{(for the result)} & \end{array}$$

Notice that we subtract when we want to find the number of years between two dates.

In the second part of the example:

$$\begin{array}{rcl} + 14 & \text{My age} & \\ - 5 & \text{When the house was built} & \\ \hline + 19 & \text{Must be obtained for the age of the house} & \end{array}$$

Here also we want the number of years between two dates; that is, we ought to subtract. Symbolically the operation would then look like this:

$$\begin{array}{r}
 + 14 \\
 - (- 5) \text{ (subtract } - 5) \\
 \hline
 + 19
 \end{array}$$

But this, so far as the final result is concerned, is the same as:

$$\begin{array}{r}
 + 14 \\
 + (+ 5) \\
 \hline
 + 19
 \end{array}$$

You will observe that when I put a + sign in front of a number to indicate the number of years *after* my birth, and when I put a - sign in front of a number to indicate the number of years *before* my birth, I have represented more than merely a number of years; that is, more than merely an amount or magnitude. I have added the idea of *direction* to the idea of magnitude. Thus + 5 does not merely represent a magnitude of 5 units, but represents 5 units *after*, or, as we shall see later, it may represent 5 units *more than*, or 5 units *above*, or 5 units *to the right of*. Similarly, - 5 does not mean merely 5 units (magnitude), but means 5 units *before* (or *less than*, or *below*, or *to the left of*).

You see, then, that the + sign and the - sign are used not only to indicate the operations of addition and subtraction, but also to indicate two *opposite* directions or senses, as *before* and *after*, *more than* and *less than*, *above* and *below*, *to the right of* and *to the left of*; and we shall find still other examples of such opposites. You will find that this new use of the + and - signs is

very convenient and useful in solving certain kinds of problems.

After this we shall call numbers preceded by such a $+$ or $-$ sign, indicating one of two opposite directions, *signed numbers*. The ones preceded by a $+$ sign, meaning after zero or above zero, we shall call *positive numbers*; those preceded by a $-$ sign, meaning before zero or below zero, we shall call *negative numbers*.

A positive or a negative number gives both a *magnitude* and a *direction*.

The magnitude represented by a signed number is called its *numerical value* or its *absolute value*. Thus the absolute value of -5 (or of $+5$) is 5. In arithmetic we use absolute values only.

This last example leads to the following rule:

To subtract a negative number gives the same result as adding the corresponding positive number; that is, the positive number having the same absolute value.

Or,

To subtract a negative number, change the sign and add.

Example 2. Two trains start at the same time from the same station. One runs at the rate of 30 miles an hour, the other at the rate of 40 miles an hour. How far apart are they at the end of two hours?

It is clear that we cannot answer the question until we know whether the trains are going in the same or in

opposite directions. Let us assume first that they are both going east, and that east is indicated by +. Then the station is the zero point on the number scale.



FIGURE 12

One of the trains is + 40 miles and the other + 30 miles from the station. The distance apart may be found thus:

$$\begin{array}{r} + 40 \\ + 30 \\ \hline + 10 \end{array} \quad \text{the result}$$

This shows only the positions of the trains on the line, and we write the result because we can perceive it. We see that we have to subtract in order to obtain 10. The symbolic operation would be shown thus:

$$\begin{array}{r} + 40 \\ - (+ 30) \\ \hline + 10 \end{array}$$

Suppose, however, that the trains had been going in opposite directions. The symbolic operation would then be slightly changed. The $-$ sign before 30 would be showing the direction, while the $-$ sign before the parenthesis would show that subtraction was required:

$$\begin{array}{r} + 40 \\ - (- 30) \\ \hline + 70 \end{array}$$

and again we should obtain the result: subtracting a negative quantity is the same as adding the corresponding positive quantity. Also the rule of subtraction is confirmed: When subtracting, change the direction sign, then add.



FIGURE 13

Example 3. Fig. 13 shows a sidewalk, with flights of steps running up and down from it. As in the time scale and the distance scale previously used, we may designate the sidewalk level by 0, the steps upward $+1, +2, +3, \dots$; and steps downward $-1, -2, -3, \dots$. We may also represent walking up by the $+$ sign and walking down by the $-$ sign.

If a man starts at the sidewalk and walks up 5 steps, *and then* walks up 8 more steps, his final position is on step $+13$. Symbolically this would be shown thus:

$$+5 + (+8) = +13$$

If he walks down five steps (-5), *and then* walks down eight more steps (-8), he would find himself on step -13 . This would be shown thus:

$$-5 + (-8) = -13$$

Observe that the words *and then* have been replaced by the symbol $+$, that is, by *addition*. Now if the man walks up five steps ($+ 5$), *and then* walks down eight steps ($- 8$) he would, of course, be on step $- 3$. Hence we write:

$$+ 5 + (- 8) = - 3$$

Similarly we should find that

$$- 5 + (+ 8) = + 3$$

Such considerations lead to the rules:

To add two positive numbers, add their absolute values and prefix the $+$ sign.

To add two negative numbers, add their absolute values and prefix the $-$ sign.

To add a negative and a positive number, take the difference of their absolute values and prefix the sign of the numerically greater number.

These rules, combined with the rule for subtraction —

To subtract any number (positive or negative) change its sign (from $+$ to $-$, or from $-$ to $+$) and add

— give all the rules necessary for the addition and subtraction of signed numbers.

CHAPTER VII

THE USE OF THE FUNCTION CONCEPT IN THE SOLUTION OF PROBLEMS

Lack of system in the traditional method of solution

THE average pupil beginning the study of mathematics by the traditional method follows directions blindly for a while. He works at problems which apparently become more and more complex. To him each problem seems different from the one before and has to be attacked in a different way. He does not feel that his skill is increasing in any definite direction. He does not see that there is any definite method or objective in view. Sooner or later he asks the inevitable question: What is the general method through which all classes of problems are solved?

Some teachers never hear this question. They have never won the confidence of the students, or they have diverted attention from word-problems through drills in symbolic manipulation so long that the pupils have come to the conclusion that mathematics is the art of juggling symbols, and that word-problems partake of the nature of puzzles, to be solved by guessing methods. This opinion is frequently shared by the teachers. They consider that a person who has attained great skill and speed in symbolic manipulation is a mathematician, even though this person is unable to

solve word-problems of moderate complexity, or to explain clearly the process used when he does solve them.

The average teacher dislikes word-problems because he has to make sure of his solutions every time he assigns them. He must also make sure that he can explain clearly the various steps if the class should ask him. This is not always easy, as some problems, while readily solved by advanced methods, are not always simple of solution by the elementary methods at the disposal of the class.

How do people solve problems?

Is there a special mental process for each problem, or is there a systematic general procedure, with definite rules and a minimum of directed mental effort?

A long investigation, during which many leading textbooks were examined, once led the author to the usual conclusion that there was not and could not be any system, and that mathematics did not have any general method through which problems could be solved. A few quotations will illustrate the usual directions for solving problems, as given in modern textbooks:

... Indicate by means of algebraic signs, the same operations on the unknown quantities as would be necessary to verify the value of the unknown quantity, were such value known....

This sounds easy, until you try it. This author un-

fortunately forgot to illustrate his method by means of examples.

... Denote the unknown number by a letter, then translate the verbal statement of the number relations into symbolic statement in equation form. ...

Evidently this writer never tried his method himself, otherwise he would have discovered that the "number relation" is frequently implied and not directly stated. Thus, the problem: At what time after 3 o'clock are the hands of a watch at a right angle, is rather difficult to solve by this method. What is the verbal statement of the number relations?

... see clearly which things in the problem are known and which are unknown. ... By a careful study of the relations between the parts of the problem, express the word statement in algebraic form. ...

This last author cautiously adds:

Sometimes this will mean an equation and sometimes not. ... Try, by reading the problem, to write down an equation. ...

Here is certainly a "gem of human thought"! This author has tersely described the "method" used by schoolboys (and many of their elders) for the last few centuries. This method was extensively used by the author in his youth. You read the problem, write down an equation, solve it, find that the result is not satisfactory, read the problem again, write down another equation, and so on, repeating the process until the right result is obtained. Then you write the steps

which you (should have) used, and finally you analyze the problem. People who have never tried the method can hardly realize how much easier it is to analyze a problem after it is solved!

But most authors are more cautious and either confess inability to give clear, positive directions, or simply omit the subject. An algebra published in 1923 does not give any method of solution. The child is left to invent a system of his own. According to the inductive theory of learning, this should be easy: generalize after learning special instances and concrete examples. But how many do?

The textbook writers, then, do not give any assistance whatever in the formation of an adequate plan of solution or of good habits of mathematical reasoning. They try to convince the children that mathematical skill can be acquired only through the solution of a large number of problems in some haphazard way. They offer exercises which give more practice in symbolic manipulation than is needed by any one not a professional mathematician, and teach symbolic operations more complex than the average engineer will ever use. Then they call this meaningless collection of useless devices mathematics, and claim that the study of it induces quantitative thinking! Will they ever learn that *thinking about quantities and their relations is a process entirely different from the manipulation of symbols?*

This lamentable condition of affairs is due to the strongly entrenched traditional method which imposes

a definite procedure. The latter begins the study of simple equation problems thus, "Express the unknown by one of the last letters of the alphabet"; and winds up, "Express the other unknowns in terms of that unknown." Under such conditions the translation of a complex problem into symbolic form is nothing short of an acrobatic performance. But this mental gymnastic feat is not needed. Neither is the skill developed through it used later, when simultaneous equations force the use of more unknowns. It will be found that, if the idea of relationship (through the function concept) is properly utilized, problems can be solved by a general form of analysis with a minimum of mental exertion and of time; while skill in solving problems increases definitely with every problem solved.

The lack of a unifying idea

Lack of any observable system in solution is, then, one of the sources of dissatisfaction and loss of interest. Much trouble is also caused by the lack of a unifying idea, of a central core around which the subject can be grouped or organized.

The National Committee on Mathematical Requirements recommends:

The one great idea which is best adapted to unify the course is that of the *functional relation*. . . . The primary and underlying principle of the course . . . should be the idea of relationship between variables. . . .¹

¹ *The Reorganization of Mathematics in Secondary Education*, p. 12.

In what way is it so valuable? How do you teach it? Modern textbooks are strangely silent on these points.

One can hardly blame the authors. Information about the function concept in elementary mathematical education is meager, rather difficult to obtain, and still more difficult to understand or use after it is obtained. A few quotations from standard authorities follow:

Teach the idea of relationship between variables.

This authority has left the small details to the teacher. All that he has done is to furnish the inspiration.

Lead the pupil to form the habit of thinking about the connections that exist between related quantities.

Again no suggestions are given about the means to such a desirable end.

Do not pass *any* instance in which one quantity is related to another without calling attention to the fact and trying to have the student "see how it works."

This last thinker is less reticent and gives more specific directions; but his directions are not very helpful to the average teacher.

The last two sentences of Chapter VII of the *Report of the National Committee* are illuminating:

It is hoped that sufficient variety has been mentioned to demonstrate [to the teacher] the existence of functional ideas throughout elementary algebra and geometry. The com-

mittee feels that if this is recognized, algebra and geometry can be given new meanings to many children, and indeed to many educators, and that all students will be better able to control the actual relations which they meet in their own lives.¹

The use of the function concept

The following pages will show:

- (1) In what way the function concept can be made the unifying idea of algebra.
- (2) How it can be explained in language so simple that the average pupil can understand it and utilize it to organize the various rules, devices, and processes around it.
- (3) That one of its chief values lies in the fact that it can be used systematically in the translation of problems into symbolic form.

Through this fact we can develop a definite and general method of solving algebraic problems. By this method problem statements can be completely analyzed and solved so far as obtaining the necessary equations are concerned, once the implied conditions are determined. In order to obtain this result, however, it is necessary to alter some of the traditional methods and to establish what may well be considered one of the fundamental devices of mathematics: the recognition and use of what we have ventured to call "characteristic formulas."

¹ *Loc. cit.*, p. 73.

An analysis of problem relations

Every problem consists of statements describing one or more "situations." Each situation in a problem is described or characterized by a formula giving the fundamental relation connecting the quantities which are characteristic of this situation. An example will make clear our meaning. Consider the following problem:

A man has 22 minutes to go to the station, a distance of 2 miles. If he takes a car which travels at the rate of 1 mile in 8 minutes, at what distance from the station can he get out and walk, if he walks at the rate of 1 mile in 16 minutes?

In this problem we have two situations: the man walking and the man riding. The quantities entering each of these situations are: distance (d), time (t), and rate (r). What we mean in this case by the characteristic formula — that is, the formula which characterizes or governs each of these situations — is the uniform rate formula, $d = rt$.

We must distinguish between two kinds of relations in problems involving more than one situation:

1. *Relations between the quantities in a given situation.* This is the kind of relation already referred to as the characteristic formula. It is never (or rarely) given in the statement of a problem. Moreover, this kind of relation is often between quantities having unlike dimensions; that is, between unlike quantities. Thus, $d = rt$, is a relation between distance, time, and

the rate, the latter being of the nature of distance per time. This is a functional relationship which governs every situation involving motion at a constant rate.

2. *Relation between situations.* This kind of relation connects like quantities in two or more situations. Such relations may be given explicitly in the statement of the problem or they may be implied therein. In the above example if we represent by capital letters (D , T , R) the quantities entering the "walking" situation and by small letters (d , t , r) the quantities entering the "riding" situation, we have, in addition to the characteristic formulas $D = RT$ and $d = rt$, the relations connecting like quantities in the two situations: the time spent in walking and riding is 22 minutes, $T + t = 22$; and the distance covered in walking and riding is two miles, $D + d = 2$. It will be observed that these relations are implied in the statement of the problem, not explicitly given.

In addition to these two kinds of relations between quantities, every problem contains statements giving the value of certain quantities. In our example, the walking rate is given as one sixteenth of a mile per minute ($R = 1/16$), the riding rate as one eighth of a mile per minute ($r = 1/8$), the total time as 22 minutes and the total distance as 2 miles. This preliminary analysis has yielded a classification which will apply to a large variety, if not to all, problems met in elementary algebra:

The statement of a problem contains:

I. Relations between quantities

1. *In a given situation — the characteristic formula,*
2. *In different situations — connecting like quantities;*
 - a. *explicitly given,*
 - b. *implied.*

II. Quantities explicitly given.

One more illustration may be useful at this point:

On looking over a daybook we find the following items: John worked 3 days and Sam worked 4 days; amount paid to both, \$18. John worked 4 days and Sam worked 3 days; amount paid to both, \$17. At what rate of pay did each work?

We have four situations, two describing John's activities and two describing Sam's. When we write $3J + 4S = 18$ and $4J + 3S = 17$, we have unconsciously used the characteristic formula: amount = rate $\times$ time, or $A = RT$; and the relation between the situations: amount earned by one + amount earned by the other = total amount. This problem is so simple that an elaborate analysis is unnecessary, but it serves to show that even to such simple problems the same principles apply.

The form $Q = NU$. We have called attention to the fact that what we have called the characteristic formula of a situation is rarely given in the statement of a

problem. The pupil is supposed to know it and to recall it when needed. It is probably for this reason that the characteristic formulas which actually occur in problems found in elementary algebras are all of a very simple nature, and, indeed, are nearly all of the same form, $Q = NU$. That is, the great majority of situations which the pupil will meet in his study of elementary algebra are governed by a formula containing three quantities and such that one of the quantities equals the product of the other two. In order to substantiate this statement, a few of the most commonly met characteristic formulas will now be examined.

1. *The uniform motion formula.*

$$D = RT$$

Here we have clearly the two quantities length and time; these two are related by R , the rate, which is a length per unit of time.

2. *The amount formula.* (Amount produced or used at a uniform rate.)

$$A = RT$$

Here the amount and the time are two of the quantities. The rate is amount produced or used per unit of time.

3. *The cost formula.* Total cost = cost of one article multiplied by the number of articles.

$$C = RN$$

4. *The interest formula.* Interest = Rate multiplied by principal.

$$I = RP$$

It is clear that many of the special formulas used in science and the arts are also of this form. Thus we have in hydraulics, $Q = AV$; in the machine shop, the cutting speed-gear-beltting formula, $S = CN$; the weight formula, the length formula, etc. In physics we have the moment of force formula, the work formula, etc.

A fundamental principle — The characteristic formula

The general principle which we desire to emphasize primarily in the preceding discussion may be formulated as follows:

Every situation in a problem is governed by a characteristic formula. The recognition of such situations and of their characteristic formulas is essential to the understanding and solution of the problem.

The use of this fundamental principle will become clearer to the reader after he has seen it applied to the illustrative problems later in this chapter. He will then perceive the great simplification in the problem-solving process and the definiteness of the method of attack which it gives. He will also perceive that incomplete realization or ignorance of the existence of the characteristic formula is the main cause of the

trouble that our children experience in their quantitative thinking. Teachers know that many children are never sure when solving a problem whether to multiply or to divide one particular number by some other number. The difficulty is due to the fact that they are not taught consciously the *three aspects* of the characteristic form, which we have cited:

$$Q = NU, \quad N = \frac{Q}{U}, \quad \text{and} \quad U = \frac{Q}{N}.$$

But, until we acquire in some way the habit of using various special instances of this form in their three aspects, until we realize that every quantity in each situation must be recognized whether or not it is mentioned in the statement, no real progress in mathematics can be expected. If mathematics is studied without a realization of this general principle, problems are solved, to be sure, but many apparently unrelated rules have to be learned, to be applied to as many apparently different cases; that is, we use mathematics in the solution of problems, but we use it empirically, without any insight into the underlying general principle. On the other hand, if this principle is learned consciously, with full knowledge of the interrelation of the various formulas and of their dependence on the general unifying idea, control over problem-solving processes is obtained in a fraction of the time needed for the disorganized method used at present. It will be found that the use of the function concept in solving

problems, through its continual use of the characteristic formula, very quickly gives a mastery over the latter.

After the children have realized that problems are amenable to this principle, that each element in the characteristic formulas plays a definite rôle in the solution of problems, and hence that problem-solution follows a definite procedure, they become more willing to drill on any kind of problem, because they discover that skill acquired in the solution of one kind of problem can be transferred or utilized in solving problems of some other type. This should be frequently emphasized by the teacher, as he will find that such a realization is of great assistance in maintaining interest.

The following illustrates the actual procedure used in introducing the function concept to a class of freshmen, about one month after the beginning of the term. The previous instruction had been part individual and part recitation, with some five-minute talks at the beginning of the periods. These lectures were immediately followed by quizzes about the topics presented. This method secured intense attention for a few minutes. The home work following this particular lecture consisted of the complete analysis of one out of three problems and translation of the statements into symbols. The symbolic solution was not required, as the class was not sufficiently advanced for some of the steps requiring the use of parentheses, the negative quantity, etc.

The function notion. Classroom presentation. To-day we shall examine a notion which is very useful in the solution of problems.

Whenever you have a perplexing situation, whenever you are stopped or delayed in some activity by a condition of affairs that you do not understand, but that you must know about in order to bring about some desired result, you have a problem.

Thus you have conduct problems, health problems, and civic problems. Your subsequent actions are determined by your selection of a plan of action out of the several suggested by the situation.

When your problem contains numbers and quantities or magnitudes such as length, weight, power, cost, and so on, it is a quantitative problem, or, as we say, a mathematical problem. Its solution is obtained by following a plan of action suggested by the statement. This statement is often the disarranged description of one or more situations.¹ This description does not necessarily give everything in the best possible shape for the mathematical solution. It may contain unnecessary material, and it may be incomplete, some additional statements being implied (hidden). But it must always contain these elements:

1. The question;
2. Some numbers and quantities;
3. A relation or relations between the quantities.

¹ See the tank-problem discussion in Chapter II.

To us the most important element is the third. Problems deal with quantities that are related; that is: (Write on the blackboard.)

The quantities described in the problem are related in some way or ways

They have something in common. They are connected with each other by means of this something, this relation. By means of this they may be compared with each other. The problem can always be rewritten into separate statements about the quantities.

These statements are of two types:

1. Some completely describe a quantity. Thus:
His rate is 10 miles per hour. $R = 10$.
2. Some describe the relation between like quantities. Thus: The distance traveled by A was 10 miles more than the distance traveled by B.
 $A = B + 10$.

Both types of statements can be translated into equations. If we have enough of these equations, the solution can generally be obtained by well-known processes.

The vital importance of the notion that quantities are related can now be seen. Let us always keep in mind: (Write on the blackboard.)

1. That the problem contains quantities;
2. That the quantities must be related;

3. That these relations can be expressed in statements;
4. That the statements can be translated into equations;
5. That some of the quantities are completely described in one statement (are given), while other statements describe the relation between like quantities;
6. That some of the relations are assumed to be so well known and obvious that they are not given in the statement.

We have, then, a definite plan in our mind when we attack a problem. The problem is already half solved!

If we cannot perceive any relations between like quantities involved in the problem, then either:

1. It is impossible to solve the problem; or
2. Some of the quantities are not necessary to the solution.

Example 1. If, in a broad jump, a boy can jump 15 feet when he takes a running start of 30 feet, how far can he jump with a running start of 300 feet?

To us the 300 feet is a useless quantity, because we know that, beyond a certain distance, the running start does not affect the distance jumped. That is the quantity 300 feet is useless because it is not related to the 15 feet. The problem cannot be solved by mathe-

matical methods because the relation between the quantities described is not known.

Example 2. If a farmer obtains a \$300 crop from one acre of fruit trees planted 20 feet apart, how much would he obtain from the same acre if the trees were planted 2 feet apart?

We know that trees only 2 feet apart would interfere with each other's growth. But we do not know to what extent this interference would affect the crop. That is, the needed relation is not given or implied in the statement nor do we know it, and we cannot solve the problem.

Example 3. What is the unit stress at the base of a brick wall 18 inches thick and 50 feet high, if brick weighs 125 pounds per cubic foot?

The unit stress is the stress per unit of surface, per square inch or square foot. The 18 inches is not a necessary quantity, is not related to the unit stress, and the problem can be solved without using it.

The problems that will be given you in this course contain all the relations necessary for solution. Sometimes they are given directly; sometimes they are implied and must be discovered. Usually restatements have to be made before these statements can be translated into equations.

We shall now solve a few problems for the purpose of

illustrating the use of the relation idea in the systematic solution of problems.

Example 1. A man has 22 minutes to get to the station, a distance of 2 miles. If he takes a car which travels at the rate of 1 mile in 8 minutes, at what distance from the station can he get out and walk, if he walks at the rate of 1 mile in 16 minutes?

Analysis: The relation idea suggests that the quantities are related. What are the quantities? How are they related? What are the units in terms of which they are described? This problem contains two situations: the man walking (Situation 1) and the man riding (Situation 2).

We find statements about distance, rate, and time. The units are miles, minutes, and miles per minute.

Statements about the quantities:

1. The distance that he walks and rides is two miles. (Relating two like quantities, distances, in the two situations.)
2. The walking rate is 1 mile in 16 minutes, or $1/16$ of a mile in 1 minute. (A complete description, belonging to Situation 1.)
3. The riding rate is 1 mile in 8 minutes, or $1/8$ of a mile in 1 minute. (A complete description, belonging to Situation 2.)
4. The time spent in walking and riding is 22 minutes. (Relating two like quantities, time, in the two situations.)

Translation into symbols:

Let the capital letters refer to the walking (Situation 1), and the lower-case type to the riding (Situation 2).

$$D + d = 2 \quad (1)$$

$$\frac{1}{16} = R \quad r = \frac{1}{8} \quad (2) \text{ and } (3)$$

$$T + t = 22 \quad (4)$$

We have used all the relations described in the statement. It is clear that, since we cannot use R and r directly in (1) and (4), we have to change them to some form that we can use. We must then go beyond the data given in the statement. We must find some connection between R and r and the other quantities. Such a connection is supplied by a formula which belongs to the situation we are talking about. In this particular problem it is the well-known uniform motion formula: distance = rate $\times$ time. This formula has three aspects:

$$d = rt, \quad r = \frac{d}{t}, \quad \text{and} \quad t = \frac{d}{r}$$

Symbolic solution:

Applying the second aspect to (2) and (3) we have:

$$R = \frac{D}{T} \text{ and } r = \frac{d}{t}$$

and

$$\frac{D}{T} = \frac{1}{16} \text{ and } \frac{d}{t} = \frac{1}{8}$$

or,

$$16 D = T \text{ and } 8 d = t$$

Substituting these values in (4) we obtain:

$$16 D + 8 d = 22 \quad (5)$$

Multiplying each side of (1) by 8 and subtracting the resulting equation from (5),

$$8 D = 6 \quad \text{or} \quad D = 3/4$$

The man may walk three fourths of a mile.

We should specially note that a statement giving a relation, before it can be reduced to the symbolic form, must deal with quantities having the same unit of measurement on both sides of the equation.

We had in (4):

Time (in min.) + time (in min.) = total time (in min.).

And in (1):

Dist. (in mi.) + dist. (in mi.) = total dist. (in mi.).

That is, we cannot have:

$$\text{time} + \text{distance} = \text{time}$$

And we cannot have an equation of this type:

$$8 \text{ hours} + 17 \text{ minutes} = 20 \text{ minutes}$$

In short: The quantities to be added or subtracted in a statement must be of the same kind and expressed in terms of the same unit. Oversight of this rule is a fruitful source of error.

The relation diagram

The reader has probably noticed the peculiar arrangement of the Equations (1) to (4). Inserting the signs $\times$ and $=$ in the proper places we obtain the "relation diagram": (Read $D = RT$ and $d = rt$ vertically.)

$$\begin{array}{ccc}
 D + d = 2 & & \\
 \parallel & & \parallel \\
 1/16 = R & r = 1/8 & \\
 \times & \times & \\
 T + t = 22 & &
 \end{array}$$

When the two characteristic formulas are placed as shown, and the diagram is read vertically as well as horizontally, the arrangement shows clearly the way in which the two situations are related and illustrates the two types of relations. That is, the diagram not only shows how the quantities are related in the characteristic formulas of their own situations, but also how they are related to the like quantities of the other situation. This diagram is, then, a complete translation of the problem into mathematical symbols. *It contains everything given and implied in the statement.* Once every quantity is related to another quantity of the same kind or completely described (given), we are ready for the well-known substitution and elimination process. On the other hand, there is little use in starting the symbolic solution before every quantity is described or related; that is, the diagram automatically indicates any missing relation. This latter is

usually found to be implied somewhere in the statement.¹

Another important feature is that the diagram, when completed, shows clearly the plan of solution. The latter is best carried out by eliminating the symbols of quantity before substituting the numerical values in the final equation. In the above problem we eliminate the r 's by substituting their numerical values in the characteristic formula $r = d/t$. After this we have a choice between eliminating either the t 's or the d 's, leaving two equations and two unknowns.

As the great majority of the situations found in our algebra problems are governed by relations of the form $Q = NU$,² the method described is practically a general method of solution. A systematic and definite procedure is followed which can be divided into the following steps:

Step 1. We examine the problem with the object of finding the quantities involved and the way in which they are related. Words like distance, time, cost, etc., immediately give

¹ See the clock problem, farther on in this chapter, for an illustration of a missing relation found through the use of the diagram.

² *Editor's note:* A caution may, however, be in order here. Care should be taken that students do not gain the impression that all characteristic formulas are of the form $Q = NU$. We need only think of the formulas for the area of a circle or square, the volume of a cube, the distance traversed by a falling body, the strength of a beam, etc., to realize that many characteristic functional relationships are not of the simple linear form. The development of the ability to think quantitatively involves the ability to recognize different forms of functional relation.

us a suggestion of the type of characteristic formula required. The function concept is recalled every time this step is made.

Step 2. We write the characteristic formulas (vertically), using capitals for one situation, and lower-case letters for the other, or letters with subscripts.

Step 3. We examine the problem again and find the quantities fully described and the statements showing the relations between like quantities in the two situations.

These steps constitute the analysis of the problem.

Step 4. We translate the statements found in Step 3 into equations, which we write in our relation diagram. An examination of the diagram shows when the problem is ready for symbolic solution. This occurs when every quantity is either described, or related to the like quantity in the other situation. If some quantity is not so related, we examine our problem again to discover the implied statement arising from the assumption that the relation is known, or is so simple that it does not need description.

This is the translation step.

Step 5. Symbolic solution by substitution and elimination.

We can see now why the search for a systematic solution of problems was useless so long as the traditional method of "expressing all the quantities in terms of one, preferably the smallest," was adhered to. This method destroyed right from the start all possibilities of organization around a central idea; of showing in what way all problems are similar. The right method can be seen to be the division into separate statements, using as many symbols as there are quantities. But even this is of little use unless a clear conception of the rôle which the characteristic formula plays is acquired.

The relation diagram is not an essential part of the method of solution. The teacher, however, will find it a valuable device in graphically showing the nature of relationships. Furthermore, its use obviates the necessity for a verbose description of each symbol. The teacher who has used the tabulation method will find the diagram easier to use, and of more value to the student.

It may be asked: Can children learn this method? The conclusive answer is that many, even of the less intelligent, have already learned it and like it. Mental tests have shown that some of these successful students have intelligence quotients a little under one hundred, while the brighter children grasp it after a single explanation followed by a few illustrations.

The preceding steps in problem-solution can be condensed into the accompanying table (see page 131).

WALL CHART

THE PROBLEM

ANALYSIS
(Based on the
function concept
that the quan-
tities described
are related)

1. Division into situations.
2. Recognition of characteristic formulas needed.
3. Writing of char. formulas vertically.
4. Division of problem into statements:
 - A. Completely describing a quantity.
 - B. Explicitly stating a relation between like quantities in the various situations.
 - C. Recognition of implied relating statements.

TRANSLATION
INTO SYMBOLS
(Symbols are
easier to handle
than thoughts
or words)

1. Symbols of quantity and of relation.
2. Simple translation of statements.
3. Location of the translated statements in the diagram.

SYMBOLIC
SOLUTION
(Based on the
Equation Law.
The same opera-
tion must be
performed upon
both sides)

1. Linear equations.
2. Simultaneous linear equations — the principle of substitution.
3. Quadratic equations.
4. Simultaneous quadratic equations.

TRANSLATING
RESULT BACK
INTO WORDS
CHECK

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It is suggested that a wall chart be made and kept in sight during the term. This would show the purpose of the detail studied at any time.

The desire for practical problems

Practical-minded boys often want to know how such a method of analysis can be applied in industrial work. The following example illustrates the similarity of solution between an academic and industrial setting:

Example 2. A contractor has 22 days to finish a contract to remove 2000 tons of rock. If he rents a steam shovel which can handle 1000 tons in 8 days, how long should he keep it, if his own shovel can handle 1000 tons in 16 days?

If we compare this example with Example 1, we find that the numbers used are the same, and that the respective quantities are related through similar characteristic formulas.

We can make the similar relating or describing statements:

1. The amount removed by one shovel added to the amount removed by the other shovel is 2000 tons.
2. One shovel removes 1000 tons in 8 days: that is, its rate of work is $1/8$ thousand tons a day.
3. Similarly the rate of the other shovel is $1/16$ thousand tons a day.
4. The time that one shovel works added to the time that the other shovel works is 22 days.

These statements can be translated into equations, a and A standing for the amounts removed by each shovel, the capital letters standing for the quantities in one situation, while the lower-case letters apply to the other.

$$A + a = 2 \quad (1)$$

$$\parallel \quad \parallel$$

$$1/8 = R \quad r = 1/16 \quad (2) \text{ and } (3)$$

$$\times \quad \times$$

$$T + t = 22 \quad (4)$$

It is obvious that this problem can be solved by exactly the same method as the preceding and will give the same numerical result, the numbers, of course, having different meanings.

It should be noted that a practical problem of this type would usually be worded as follows:

A contractor has to remove 2000 tons of rock in 22 days. His steam shovel can remove $62\frac{1}{2}$ tons a day. He rents another shovel which can remove rock twice as fast. As the excavation is narrow, he can use only one shovel on the job. How much will he have to pay for the rent of the shovel at \$50 per day, if he uses his own shovel as much as possible?

An examination of this statement reveals one of the sources of trouble in "practical" problems. The question about the total cost is misleading. It obscures the real question, which is about the number of days required. It introduces another problem into the first: If it takes n days for the rented shovel, and the cost is

\$50 a day, what is the total cost? Such a setting makes the problem sound business-like; connects mathematics with life — and beginners find it almost impossible to solve.

That the last statement is true was demonstrated once when the academic setting and the last setting were assigned together as home work to a class trained by the traditional method. Most pupils solved the first, but not one succeeded in solving the second! ¹

We may give still another setting of the same problem.

Example 3. A tank has to be filled in 22 hours. One pump can fill it in 8 hours. Another pump can fill it in 16 hours. How long should the faster pump be worked if it is desired to work it as little as possible, and if only one pump can be used on the job?

A comparison with the two preceding examples will show that the methods of solution are identical.

The resetting of practical problems

The problems examined suggest an idea: If a problem is too complicated, contains too many technical terms, restate it in some better-understood setting, using the same numbers and the corresponding ele-

¹ This problem was proposed to *School Science and Mathematics* readers, and its solution discussed in the June, 1924, number, page 637. Only two of the many answers received were correct!

ments in the respective characteristic formulas used. The following example illustrates the procedure:

Example 4. Two contractors are building a road, each building one half, and each starting from one end. A has a concrete mixer, which mixes enough concrete per batch for 1 yard of road, while B's mixer gives enough for 7 feet of road; but B's mixer only gives two batches during the time that A's mixer gives three. A has a start of 60 yards. Working continuously, they finish at the same time, each having done his half of the road. How long did it take the first man, if he mixes one batch every 10 minutes?

This is a very complicated problem. It would take a good deal of time to analyze it and to evolve a plan of solution. We should have trouble in visualizing the relations between the various quantities. Let us, then, adopt the above suggestion: restate the problem, using familiar situations.

Example restated. A boy, pursued by a policeman, has a start of 60 yards. He takes 3 strides to the policeman's 2, but 3 strides of the policeman cover the same ground as 7 strides of the boy, who covers 1 yard in each stride. How many yards (boy's strides) must the policeman run to overtake the boy?

This statement is far easier to visualize. We understand clearly every separate condition. Yet a comparison with the contractor problem shows that identically the same numbers were used with the correspond-

ing quantities and similar relations, both in characteristic formulas and in the two situations described in each statement. The number of yards that the boy will run is equal to the number of yards made by the first contractor. On the other hand, the second statement is of such a nature that the quantities and their relations are in the foreground; the policeman and the boy occupy a secondary position when we try to keep a view of the problem as a whole in our mind, while forming a plan of solution.

Analysis. Two situations are described: the boy running, and the policeman running. The situations are related by the fact that the time elements involved are equal (this is an example of an implied statement), and by the fact that the distance run by the policeman is 60 yards greater than the distance run by the boy. Rate of stride relations are also given.

The quantities described are distances, rates, and time. These suggest that the characteristic formula needed is the uniform motion formula.

The following relating and describing statements can be made:

1. The boy covers 1 yard, or 3 feet, in 1 stride.
2. The policeman covers the same amount of ground in 3 strides as the boy covers in 7, or 21 feet. That is, the policeman has a 7-foot stride.
3. Two strides of the policeman, 14 feet, and 3 strides of the boy, 9 feet, are made in the same

length of time. That is, the distance covered by the boy in a unit length of time — his rate — is $9/14$ that of the policeman.¹

4. The distance run by the policeman = distance run by the boy + the boy's start.

Statements 1 and 2 were combined to form Statement 3, so no equation derived from them is needed. We translate Statements 3 and 4 into symbols, together with the implied statement that the time element is the same in both situations.

$$D = d + 60 \quad (1)$$

$$\parallel \quad \parallel$$

$$R = r \times \frac{14}{9} \quad (2)$$

$$\times \cdot \times$$

$$T = t \quad (3)$$

The capitals refer to the policeman, while the lower-case letters refer to the boy.

Substituting the value of rt obtained from (2) into (1),

$$D = 9 D/14 + 60$$

$$5 D/14 = 60; \text{ or } D = 168$$

The policeman runs 168 yards. The boy runs 108 yards.

¹ At this point a person skilled in the perception of relationships may introduce a short cut: Since the time elements are equal, $5/14$ of the total distance is 60 yards. But any one who has met the hare-and-hound problem in algebras knows that such a mental process is beyond the power of a beginner.

Degrees of difficulty in problems

Uniform rate problems, in which amount of work done enters as an element, are very often met in industry. Owing to the complexity of the statements and to lack of sound mathematical training, direct solutions are usually avoided and guesses are made. But the complexity of the situations often disappears when other problems are made up which use some other instance of the characteristic formula, as has been shown in the contractor-policeman problem. Problems can be divided into four classes:

1. They may have simple settings and simple relationships;
2. They may have simple relationships and complex settings;
3. They may have complex relationships and simple settings;
4. They may have complex relationships and complex settings.

It is rather astonishing that writers who devise tests of ability to solve problems never grade problems from this point of view. The relation diagram shows at a glance just how difficult the problem is from the point of view of relationships. On the other hand, psychologists have considerable information about the vocabulary of school children and their ability to understand meanings. A test taking both elements into account could be devised without much difficulty.

We shall now proceed to show, by means of further illustrations, how situations in which work, for example, performed at two different rates can be compared to situations in which trains or automobiles travel at different rates. Our aim is to demonstrate: (1) that rate problems of any kind, in any setting, are highly practical; (2) that the study of the relations between quantities and of the methods of solution is more important to the learner than the industrial terms which make the problem appear practical, but which really obscure the solution and retard the development of power in quantitative thinking; (3) that the skill acquired in solving any kind of problem can be transferred to the solution of practical problems, provided the underlying principles are known and appreciated.

Again we repeat: Children develop capacity for quantitative thinking by handling problems in which quantities and *their relations* are recognized and consciously kept in the foreground, the complexity of the setting being gradually increased. They cannot develop such capacity when they learn to solve problems by means of rules and directions imposed from without, especially when these problems are complicated from the start by a mass of irrelevant detail.

Further illustrative examples

Example 5. Two gangs of sidewalk builders start from the same end of a street on opposite sides. One

gang makes 12 yards of sidewalk more than does the other every day. The faster gang reaches the end, 240 yards away, and turns back at once on the other side of the street, meeting the slower gang 192 yards from their common starting-point. Find the time taken.

Example rewritten. Two automobiles start at the same time from a certain town. One travels 12 miles an hour faster than the other. The faster machine reaches another town 240 miles distant and turns back at once, meeting the slower one 192 miles from their common starting-point. Find the rate of each.

The two situations in this problem are related by the times being equal, and one rate being 12 miles per hour greater than the other. The quantities are distances, times, and rates. The units are miles, hours, and miles per hour.

Statements describing or relating the quantities and situations:

1. The slow machine travels 192 miles. (A complete description, belonging to Situation 1.)
2. The faster machine travels 288 miles. (A complete description, belonging to Situation 2.)
3. The rate of the faster machine is 12 miles an hour greater than the rate of the slower. (Relating two corresponding quantities — rates — in the two situations.)
4. The time of traveling of both is equal. (An implied statement, relating two corresponding quantities — times — in the two situations.)

5. The characteristic formula is the constant rate formula for distance ($d = rt$).

Translation into symbols. (The capital letters relate to the slower machine and the lower-case letters to the faster.)

$$192 = D \quad d = 288 \quad (1) \text{ and } (2)$$

$$\parallel \quad \parallel$$

$$12 + R = r \quad (3)$$

$$\times \quad \times$$

$$T = t \quad (4)$$

Dividing (2) by (1),

$$\frac{rt}{RT} = \frac{288}{192} = \frac{3}{2}$$

Substituting the value $T = t$ from (4), and multiplying by $2 R$,

$$2 r = 3 R$$

Substituting this value in (3),

$$2 (R + 12) = 3 R, \quad \text{whence } R = 24$$

Translating into words, the rate of the slower machine is 24 miles an hour. This result can be checked by seeing that it satisfies the conditions of the statement.

After a keener perception of relationships has been developed, this problem may be solved as follows:

1. The faster machine travels 96 miles more than the slower.

2. Since it travels 12 miles an hour faster than the slower machine, the time required is $96/12 = 8$ hours. The rate of the slower machine is $192/8 = 24$.

Such solutions are gravely given as illustrating "methods of solution," and children are often berated because they fail to "learn" them. But teachers know from bitter experience that logical reasoning processes are not learned by imitation. Their problem is to teach the art of thinking, and not how other people have thought in former times.

In designing or adjusting machinery, a problem of the following type frequently arises: Two parts of the machine start from different points at different times, and move at different rates. At what time and in what position will they be at some given distance apart? It would be of little use to the student to examine a specific case of this type; first, because the statement would be very complicated, and hence would require a large amount of time without a proportionate return of usefulness, and secondly, because he may never meet this particular problem in practical experience. It is, however, very useful to learn the general method of solution of such problems; this is best done by using an easily visualized situation, dealing with well-known quantities. The following problem contains all the elements of the machinery problem suggested above.

Example 6. A bicycle rider starts from A toward B, 200 miles away. His rate is 9 miles an hour. Three hours later an automobile starts from C, which is 20 miles on the other side from A, passes through A, reaches B, where it stops for an hour, then returns toward A; how soon after leaving B will the automobile be 15 miles from the bicycle rider, if its rate is 26 miles an hour?

This problem is easily visualized and the relation diagram written down. Let x be the distance that the automobile traveled after leaving B.

$$\begin{array}{rcc}
 200 - 15 - x = D & d = x + 20 + 200 \\
 \parallel & \parallel \\
 9 = R & r = 26 \\
 \times & \times \\
 T = t + 4
 \end{array}$$

x is eliminated by comparison, since $x = 185 - D$, and $x = d - 220$, and we get the final diagram:

$$\begin{array}{rcc}
 185 - D = d - 220 \\
 \parallel & \parallel \\
 9 = R & r = 26 \\
 \times & \times \\
 T = t + 4
 \end{array}$$

The rest of the solution proceeds as follows:

$$\begin{array}{rcl}
 185 - D & = & d - 220 \\
 185 - 9T & = & 26t - 220 \\
 405 - 9t - 36 & = & 26t \\
 369 & = & 35t
 \end{array}$$

Whence $t = 10\frac{2}{5}$ hours.

Clear thinkers may adopt the following short cut:

After the bicycle rider has traveled 4 hours — that is, covered 36 miles — the two men approach each other at the rate of 35 miles an hour. The sum of the distances (traveled by both at the same time) is:

$$200 + 220 - 15 - 36 = 369 \text{ miles,}$$

and the time is $369/35$.

The unprejudiced reader must acknowledge the superiority of the method shown over the traditional method. Many algebraic problems are practically impossible of solution without a systematic analysis by statements, which the described method easily supplies. We can easily find the statements describing or relating the quantities once we know what to look for. But we know what to look for only if we keep the idea of relationship in mind, as the National Committee on Mathematical Requirements recommends. The particular form of the characteristic formula required is usually apparent at once from the words used in the statement. These two, then, are the fundamental notions needed for systematic problem-solving.

If the notation is carefully observed, translation into symbols does not present any particular difficulties. It has been found that children learn this part of the solution very quickly. After this the symbolic manipulation is clearly indicated, and can usually be carried out in more than one way. It is evident, then, that,

for a large class of problems, we have here a general method of solution.

The plan of solution of the last problem is easily applied to one of the traditional problems, the clock problem. This problem illustrates the incomplete statement sometimes present, and also shows how the notion that quantities are related through the characteristic formula assists in the automatic recall of the missing relations.

Example 7. At what time after 3 o'clock will the two hands of a clock be at a right angle to each other?

We have two situations, the two hands moving at different rates. The quantities described are the angle (or length of arc) between the two hands, and the time of starting. The statement is very brief because it is assumed that every one knows the conditions under which a clock runs. We assume that the unit of time is the minute, the unit of arc the minute-space, and we observe that the hands start moving at 3 o'clock (just as an automobile and a pedestrian start from different places at the same time) and that at the end they are to be 15 minute-spaces apart.

Statement about the quantities:

1. The time during which the two hands move is the same for both.
2. The arc covered by the minute hand is the sum of the 15 minute-spaces necessary to reach the

figure 3, the distance traveled by the hour hand, and 15 minute-spaces besides.

Translation into symbols:

$$D = d + 15 + 15$$

$$\parallel \quad \parallel$$

$$R \quad r$$

$$\times \quad \times$$

$$T = t$$

It is evident that, since R and r are neither given nor related, we cannot solve the problem. There must be a statement implied, relating or describing the rates. This statement is: The minute hand moves at the rate of one minute-space a minute, and the hour hand moves at one twelfth of this rate. That is $R = 1$, and $r = 1/12$.

This enables us to complete the relation diagram.

$$D = d + 30 \tag{1}$$

$$\parallel \quad \parallel$$

$$1 = R \quad r = 1/12 \tag{2 \text{ and } 3}$$

$$\times \quad \times$$

$$T = t \tag{4}$$

Substituting the values of D and d in (1),

$$RT = rt + 30$$

$$T = t/12 + 30$$

Since $T = t$

$$\frac{11}{12} T = 30; \text{ or } T = 32 \frac{8}{11} \text{ minutes.}$$

That is, the two hands will be 15 minute-spaces apart, or at right angles, at 3 hr. $32\frac{8}{11}$ min.

It should be noted that all the usual clock problems can be solved by using the last diagram. The only change due to different conditions will be in the number 30. That is, the relative position of the hands is automatically followed by a corresponding change in the statement about the distance covered by the minute hand, in Equation (1).

Quadratic equations

As it is not intended to discuss quadratic equations very extensively, the following example will merely illustrate the application of our method of solution to this often difficult class of problems:

Example 8. A certain bar of iron weighs 36 pounds. If the bar had been one foot longer, each foot would have weighed one half pound less. Find the weight of the bar per foot and its length.

The characteristic formula takes the following form in this two-situation problem:

Total weight = weight per foot $\times$ number of feet.

$$T = WN$$

We can make the following statements relating or describing the quantities:

1. The weight remains unchanged (implied), and is 36 pounds.

2. The length is increased one foot in the second situation.
3. The weight per foot is diminished one half pound in the second situation.

Translation:

$$T = t = 36 \quad (1)$$

$$\parallel \quad \parallel$$

$$N = n + 1 \quad (2)$$

$$\times \quad \times$$

$$W = w - \frac{1}{2} \quad (3)$$

Substituting T , N and t , n for W and w in (3), we have:

$$\frac{T}{N} = \frac{t}{n} - \frac{1}{2}$$

Simplifying:

$$nT - tN = -\frac{nN}{2}$$

Substituting the values given in (1) and (2),

$$36n - 36(n + 1) = -\frac{1}{2}n(n + 1)$$

Which reduces to the quadratic equation,

$$n^2 + n - 72 = 0$$

Solving this, we obtain $n = 8$ feet.

The reader will find more complex problems ana-

lyzed in Chapter VIII, including problems leading to cubic equations.

Criticism of the method

The method described is young, but it has already run the gauntlet of adverse criticism: offered mainly by educators who have not given it a trial. We quote from one conservative:

This diagram method makes problem solution so easy, so systematic, that the disciplinary value of the study of mathematics is lost. It kills analysis. It provides a mental slide-rule. By it children are enabled to solve problems without thinking, or at least without doing the severe mental work which is good for them. It should be carefully examined by experts for a few years before being published . . .

Thorough acquaintance with the method, and analysis of the results obtained in classroom work, enable the author to answer these objections as follows:

1. The so-called analysis popular at the present time has no scientific basis. It depends for success upon a mixture of apperceptive background, common-sense, native shrewdness, recollection of problems solved in the past, and patience in trying various equations until the right solution is accidentally obtained. While it is but fair to say that some modern authors have given directions for solving problems systematically — for instance, by tabulation, or by writing down statements — these methods are illustrated only through simple problems leading to linear equations, such as any bright student could write down readily after reading

the problem. Furthermore, the tabular or statement and reasoning methods are far more cumbersome than the method described, and do not provide the conscious and constant recall of the function concept given by the latter.

If the method described kills the traditional analysis, it replaces it with a different kind of analysis of far higher educational value. The pupil is trained in minute observation of exact meaning of words and sentences, in paraphrasing until he obtains a sentence that can be translated into an equation. But, most important of all, he is trained in watching for and discovering relationships, not once, but several times in each problem. The teacher does not need to "point out relationships whenever they occur" (*National Committee Report*), the pupil finds them himself, and *uses* them.

What this critic called analysis is only the orderly arrangement of steps and *post-mortem* examination held after a problem is solved. No one ever analyzed by pure logic a difficult problem that he had never before met. On the other hand, the author has often seen unfamiliar problems analyzed systematically by children using his method. The problems were solved in such a short time that it was evident that the steps were taken consecutively and no guessing was done. Thus, one of his pupils analyzed twenty-five rate problems correctly in ten minutes, at times filling in the relation diagrams as fast as he could read the statements.

2. The fact that ordinary problems given in an elementary course become too easy when the function method is used should not be a reason for rejecting the method. More difficult problems are easy to provide. That is, more ground can be covered in the same length of time. This should effectively answer the "destroy the disciplinary value" and "severe mental effort" arguments.

3. Nine tenths of all students taking a ninth-grade course will not go to college. To them mathematics becomes significant only after they become convinced that it can be useful after they leave school. The definition of mathematics as being the art through which problems dealing with quantities can be solved is the only one which satisfies this class of students. To them mathematics must be shown to have a social value before they can be induced to apply themselves energetically to it. Ignoring the rights and needs of these students is one of the causes of the present dissatisfaction with high-school mathematics.

4. Analyses of the results obtained from the traditional teaching of the first year of high-school mathematics are anything but satisfactory. Pupils have learned to solve equations, but not problems. Truthful pupils will often acknowledge that they solved problems by guessing the answer, then working backward in some haphazard way to the equation, or that they solved by arithmetic methods whenever it was possible. The reader is probably well acquainted

with these devious and highly reprehensible practices.

Motivating features of the method

The chief value of the method described here lies in the fact that through it the child is enabled to solve problems almost from the beginning of the term, and notices that his power of attacking problems increases rapidly. The usual result is that he quickly becomes impatient at his own lack of manipulative skill and voluntarily drills in the mechanical parts of the subject with an earnestness unbelievable to a teacher using the traditional method. It is needless to comment on results obtained under such conditions.

Need of reorganization

We have divided the solution of problems into four main steps:

1. The analysis of the statement;
2. The translation into symbols;
3. The manipulation of the symbols;
4. The interpretation of the result.

If we examine the ninth-grade textbooks used in our schools, we find that what is called mathematics is mainly a more or less logical arrangement of descriptions and dissertations concerning the third and fourth steps. Many authors and teachers are still of the opinion that training in symbolic juggling creates abil-

ity to solve problems. They seem to consider problems as little diversions which should not distract attention from mathematics proper.

But the public knows from long experience that this symbolic manipulation leads nowhere in the case of most students. One year of it has no practical value whatever. Hence the protests, hence the gradual elimination of mathematics as a requisite for graduation from high school. Fifty years ago mathematics was one of the main subjects of the curriculum. At the present time it is in danger of occupying a secondary position and of losing ground every year. And what is being done to remedy this condition of affairs, to restore mathematics to the place that it should occupy? Only rearrangement of the contents, addition of what some adult considers interesting material, and elimination of other parts considered less interesting. But the main point is overlooked: you cannot teach symbolic manipulation effectively before the pupil knows what it is all about and what it is good for, and you should not teach equations before he knows how equations are obtained.

Principles underlying the reorganization

Mathematics must be reorganized from the problem-solving point of view. The equation must be derived before the need of the symbolic solution is perceived. The end must be known before the means are discussed or studied. Teaching symbolic solution before its

need is perceived may be logical, but like many other logical methods it fails because the human element is overlooked. It may be intended psychologically, but psychologists must surely know that no child can keep up his interest month after month unless he knows that he is progressing, is mastering a useful subject, and sees the end in sight. A subject so organized that the end is remote and indefinite, or even not in sight, is bound to produce such results as we are generally obtaining from ninth-grade mathematics at the present time.

When manual training was first introduced into the schools, the regular teachers took hold of the instruction and attempted to teach the subject logically and pedagogically. The pupils were first requested to memorize the names of tools and to be able to describe their uses and the methods of handling them. Then they were given exercises which were supposed to develop certain skills with some tools. They had to become proficient in tool-handling before they were allowed to make any useful articles. The results obtained were so poor that this method was soon thrown on the scrap-heap. Manual-training teachers are now practical mechanics who show their pupils how to make useful articles from the start. The pupils learn the names and uses of tools as they go along and develop skill from the actual use of tools in real work.

Why cannot mathematics be taught in the same way? Let the pupils try their skill on problems from

the very start. Show them briefly how to analyze and translate into symbols. Teach them bits of symbolic manipulation as they go along, only as they need them. Help them to finish their problems, just as the shop teacher adds the finishing touches to his pupils' work. It will not be long before they become impatient at their lack of skill and ask for drill in manipulation.

The equation, the negative quantity, and the parenthesis are easily mastered when used as tools in problem-solution in the first semester. On the other hand, when learned as separate units with "illustrative examples" of their use, they fail to arouse interest. It is remarkable that psychologists have overlooked this method of motivating the study. They have paid entirely too much attention to attempts to make a competitive game out of it and to other artificial ways of causing "satisfaction." But the "neurone-tickling" process is pretty well discredited now, meaningless drills are still nothing but drudgery, while the skill acquired by such artificial methods is easily lost. Then we resort to reviews and reviews of reviews, in vain attempts to create habits of quantitative thinking without ourselves having clear ideas about the nature of relationships among quantities.

Essential features in the reorganization

Elementary algebra must, then, be reorganized along the three following main lines:

1. Problem analysis must be given its proper place in the scheme of instruction.
2. Symbolic manipulation must be restricted at each stage to the types and to the complexity needed to solve the problems.
3. Sequence of subject-matter must be determined by the number of situations involved in problems and the nature of the relations between the quantities in these situations. Toward the end of the course, problems should become more and more practical; that is, contain a setting of increasing complexity from the point of view of technical or commercial terminology.

Such an arrangement would satisfy everybody with the possible exception of the college authorities. But, when it is considered that a college-bound student will take another year of algebra and one year of geometry, we can see how a clear understanding of the methods and uses of mathematics, a vivid perception of the derivation and use of symbolic notation, and a comprehension of the function concept will eventually more than make up for any lack of facility in symbol-juggling. Furthermore, it is already noticeable that college officials are beginning to see the light. They are beginning to change their obsolete entrance requirements, and to recognize that problem-solving ability and not manipulative skill is the better criterion of mathematical preparation. They are just beginning to learn that students trained in schools where

small regard is paid to college-entrance requirements often succeed better in later academic work than those trained by regulation college-preparatory methods. The former learn the mathematical language; the latter attempt to learn only the shorthand of this language. The former learn to handle thoughts; the latter learn to handle only symbols of thoughts.

CHAPTER VIII

THE ANALYSIS OF PROBLEMS

Present methods do not teach analysis

By a gradual process of evolution we have arrived at the present method of teaching elementary mathematics, which apparently aims to show the child how to use mathematics in life work, but does not show why certain fundamental processes are used, nor when to use them. The child learns how to multiply or divide, but not when to multiply or divide. He learns to solve equations, but not how to obtain equations.

The child learns rules through highly developed psychological methods. After these rules are learned, they are seldom applied to problems because the analysis which would show *where* these rules should be applied has not been taught. The next step in these methods is the haphazard solution of a large number of problems at break-neck speed, leading to the gradual acquisition of skill and experience for the few survivors. Learning mathematics according to present methods has been described by an "authority" as a process of "solving such a large number of problems that a new one is recognized as belonging to a certain type already solved."

If this were true, we should cease to wonder that so few people can solve problems, and also cease to teach

mathematics to that great majority who will not solve thousands of problems in the course of their training.

But it is not true. It has been shown that the chief difficulty is due to lack of knowledge of essentials and lack of system in analyzing problems. It has been shown that problem-solving is comparatively easy once the general structure of problems is understood. We have analyzed a few problems dealing with ordinary situations and given a general method of attacking well-known types. It is now our purpose to extend this analysis a little farther and to show how problem-solving can be taught to children who have learned long division, fractions, and the simple equation. After this we shall examine the practical industrial problem.

Elementary problems. Classroom presentation. Before we can play baseball we have to learn to throw the ball and to bat. After this we have to learn the rules and methods of playing. The best batter, runner, and catcher of balls on earth could not play baseball unless he knew the rules and knew what to do after catching a ball or hitting one.

The same is true of mathematics. The pupil who can work the largest number of exercises in division or equations and do them correctly may be unable to solve simple problems because he does not understand, and hence cannot apply, the rules that must be observed in the solution of problems. These rules are

best learned by analyzing problems before solving them, breaking them into parts. We find two elements in most problems:

1. The mathematical element, consisting of numbers, and words necessary to the solution. These describe the situation in such a way that we can find some unknown quantity.
2. The setting, consisting of words that are intended to make the problem interesting and to make you understand that you are learning something useful.

But the moment you begin to solve problems, you discover that the settings are in your way. Their interesting aspects distract your attention from the essential points, the points which you must combine and manipulate in order to solve the problems. You find that in order to obtain solutions you must actually strip the problems down to the essential elements, reduce them to what may be called the mathematical words.

After you have done this stripping, you may still have a deal of trouble in deciding what to do next. You should then examine the rewritten problems to see how many different situations are described and how many quantities are involved.

Example 1. A man walked twelve miles in four hours. How fast did he walk?

This is a one-situation problem. Two quantities are completely described: a distance and a period of time. The question is about a third quantity which is *related* to the other two by the rule: rate = distance divided by time; that is, by the formula $r = d/t$. To solve the problem, we write 12 instead of d , and so on.

Example 2. Twelve shovels weigh sixty pounds. What is the weight of one?

This is a one-situation problem. The quantities fully described are the number of articles and the total weight. The question is about a third quantity which is *related* to the other two by the rule: weight of one equals total weight divided by number of articles; that is, by the formula $w = W/N$; and so on.

Example 3. How long a board is needed if seven pieces each two feet long are desired?

The quantities are the number of pieces and the length of each piece. The question is about the third quantity, the total length. The formula required is $T = LN$.

You should always keep in mind the existence of the third factor *in each situation* when solving problems involving multiplication or division. It is always supposed to be present. Your progress in problem-solving depends largely upon your ability to recognize it and to state it when it is not given directly.

Discussion. Exercises of this type develop ability to recognize the quantities and characteristic formulas involved in one-situation problems. They establish the fundamental notion of problem structure.

It has been repeatedly stated here that the failure to recognize clearly all the factors involved in problems requiring multiplication or division is the main cause of our early quantitative troubles. This failure is usually due to the desire to save time and cover a large amount of ground in early school work. While the objectives are clearly established, the means needed are more than hazy. We are trying to train children to recognize relations among quantities without telling them what we want them to do, without telling them specifically what the quantities are, and without pointing out clearly the nature of the relationships that we want them to see.

Thus the problem: What is the cost of one hat if twelve hats cost \$36? is usually solved by dividing 36 by 12, without a thought about the reason for the step. Many children will say that they did it because 36 is a larger number than 12. Such children would also divide 36 by 12 if the problem were: What is the cost of one hat if thirty-six hats cost \$12? Few teachers ask children how they solve problems. They are only interested in results, and consider that a process is learned when correct answers are given. But the unsoundness of the training becomes apparent as soon as the child is given a problem of the following type:

147 yards of rope cost \$4.67; what is the cost of 43 yards?

If able to solve it when the elements are presented in the above order, the child will often fail when the sequence is varied, as in:

What is the cost of 43 yards of rope if \$4.67 is the cost of 147 yards?

Careful observation has shown that a great many children will try manipulation of the three numbers until they find a chance answer that appears reasonable. It has also shown that *most* children acquire the notion that problems *must* be solved by using the numbers given in the problem, and that the only permissible combinations are obtained through the use of the four "fundamental" processes. That is, the notion that relationships are often implied is completely lacking.

This deplorable condition of affairs is due to the mistaken practice of: (1) Introducing short cuts before the child is taught to analyze or even to perceive the elements of the problem, and before he has a chance to learn that quantities are related and the nature of the relationships; (2) giving the high-speed drills devised by efficiency experts aiming at quantity production.

It has been found in industry that efficiency experts often overlook some essential process. We find in education that our experts frequently overlook the fundamental fact that thinking requires time and leisure.

We cannot associate and organize unless we are given a definite amount of time. Mental processes are slow in adults; in children they are still slower.

The slow method

In the first problem, "If twelve hats cost \$36, what is the cost of one?" the child should be taught to recognize and state the three quantities involved as factors of the characteristic formula $C = RN$. Substituting the values given, $36 = 12 R$. Translating back into words: Twelve times the cost of one is thirty-six. What is the cost of one? $R = ?$ The child thinks and sees the symbols of thought at one and the same time. The thought gives meaning to the symbols, and the symbols assist the thought in taking a definite direction. The information has been associated and organized.

The unitary reasoning

Such a method would also develop power in unitary reasoning, the only short-cut device of any value from the pedagogical and the practical point of view. For some unknown reason this device is no longer taught, being replaced by "psychological" methods which proceed "from easy to difficult" by first providing practice in easy examples, gradually accelerating the process until top speed is obtained, then abruptly jumping into two-situation problems in which the main relationship is implied. These methods trust to rules,

drills, and ingenious habit-forming devices to develop thinking about quantities without mentioning quantities. Troublesome explanations about fundamentals are left out in the hope that children will form methods of their own inductively. Few do so.

Objections to proposed methods

It may be objected that the methods described in this book are unnecessarily profuse in trivial details; that a relatively unimportant process or principle can be learned faster by "psychological" methods; that the method described is not practical; that the child needs practical drills in short cuts or short methods, not long-drawn-out analyses; that this method cannot be used in business, as it is too slow.

The only answer needed is that this method is fundamental. It gives a process essential to clear understanding and future growth. Short cuts partake of the nature of blind alleys. The user of short cuts gets into a frame of mind in which he solves drill problems without thinking, and sees no need of using analytic methods (quantitative thinking) for the solution of problems which can be solved by means of short cuts. He does not realize that practice in the longer reasoning methods is imperative if complex problems, beyond the scope of the short cut, are to be attempted later. There is no transfer of training in reasoning to other studies, because rules do not require reasoning. Our numerous mathematical failures are due to the lack of

development of mathematical reasoning in the early grades; they are due to methods which can only produce inhibition of original thinking. We are too easily influenced by the impractical demands of self-styled "practical" people that children be taught something tangible, not theorizing (that is, not thinking).

The reasoning of these efficiency experts runs somewhat as follows: "To solve a problem by analytic methods takes ten times as long as by short-cut methods. Therefore the short-cut method is the best." But results speak for themselves: few children become proficient in short-cut methods, and the rest cannot solve problems at all, if the complaints of business men and college professors is a criterion. When a problem is met in life work, the main objective is to get the correct solution; the length of time used in the process is usually of little relative importance. On the other hand, a person who has to solve a large number of problems of one sort finds little difficulty in devising or looking up short cuts after the first few are solved by analytic methods. Is it not preferable that children should be slow in solving their problems, rather than that they should not be able to solve them at all?

The temptation to teach "practical" short cuts is almost irresistible to many teachers. But careful analysis shows that most short-cut methods omit at least one step in reasoning, or require the perception of some implied relation utterly beyond the capacity of

the learner. This at once makes the use of the short cut conditional upon the recall of some memorized rule and upon the recognition of the feature which makes the problem possible of solution by that particular short cut. Is this practical?

Classification of problems

It is suggested that problem-solving be taught in the following sequence of topics, determined by the complexity in the "structural" relationships between the quantities:

1. One-situation problems, with an analysis of the more commonly used characteristic formulas, pointing out the resemblance between them, and the properties of the factors.
2. Two-situation problems, in which the situations are not related. The solution of such problems is obtained by the comparison of like quantities.
3. Two-situation problems, in which the situations are so related as to lead to the equality of two ratios, that is, to proportion problems.
4. Two-situation problems in which the relations do not lead to a proportion, solved by simultaneous equations.
5. Two-situation problems solved by quadratic equations.
6. Problems with three or more situations.
7. Problems with technical settings, requiring a process of translation into mathematical language.

We shall now proceed with a brief analysis of problems falling into each of the above groups.

One-situation problems

We first note that one-situation problems, excepting those dealing with fractions or parts of some quantity, usually contain two quantities and the factor showing their relation, or, as we assume for convenience in our discussion, of three quantities. Then the characteristic formula needed for the solution can take any of the three forms:

$$Q = NU, \quad N = \frac{Q}{U}, \quad \text{or} \quad U = \frac{Q}{N}$$

Thus in the uniform motion problem, we have $d = rt$, $r = d/t$, and $t = d/r$. That is, we can have only three types of problem:

1. Given the time and the rate, find the distance.
2. Given the distance and the time, find the rate.
3. Given the distance and the rate, find the time.

The teacher should point out the similarity between the uniform motion formula, the general constant rate formula, and whatever characteristic formula happens to be involved in the particular problem examined. If this policy is adhered to consistently, the child will form a habit of looking for the characteristic relation in every problem presented. Some day, when confronted by a difficult problem involving some little-

used form, he will consciously compare it with some well-known and well-understood form. When he does this, we may consider that he has acquired sound habits of quantitative thinking.

One-situation problems, when met in elementary textbooks, usually require the use of one of the following characteristic formulas:

1. The uniform rate formula for work: $A = RT$.
2. The uniform rate formula for distance: $D = RT$.
3. The cost formula: $C = RN$.
4. The weight formula: $T = WN$.
5. The length formula: $T = LN$.
6. The interest formula: $I = PRT$.¹

Two-situation problems comparing like quantities

1. A man purchases a bond for \$279, and receives \$17.80 interest in one year. His friend purchases another bond for \$380, and receives \$24.63 in the same period of time. Which bond pays the higher rate?
2. A farmer rents 47 acres and pays \$350 a year rent. His neighbor pays \$510 for the rent of 67 acres of the same kind of land. Which pays the higher rate?
3. A boy batted 57 hits out of 147 times at the bat.

¹ The last is a "compounded" characteristic formula. We may show the compounding as follows: The interest for one year is $I_1 = PR$. This is one characteristic relation. The interest for T years is $I_T = I_1 T$. This is another characteristic relation.

Another boy batted 72 hits out of 170 times. Which is the better batter?

4. One kind of coal costs \$10 a ton. It contains 8 per cent ashes. Another kind contains 11 per cent ashes, but its cost is only \$9 a ton. Which is the cheaper?
5. Three varieties of wheat are planted in separate plots of ground in the same field; 350 pounds of seed of the first kind produce 7860 pounds harvest; 490 pounds of the second kind produce 9765 pounds; and 264 pounds of the third kind produce 5142 pounds. Which is preferable?
6. A cow gives 40 pounds of milk containing 3.1 per cent butter fat. Another cow gives 32 pounds of milk containing 4 per cent butter fat. Which gives the more butter?

Analysis shows that these problems contain groups of unrelated situations. We solve each characteristic formula for its unknown factor, and compare the results in pairs. They are valuable because all the quantities are in plain sight, and every factor of the characteristic formula must be used.

Problems of this type are valuable because they show the utility of mathematics in everyday life situations. A farmer compares returns from seeds, fields, or stock. The investor compares the returns from various pieces of property. The housewife compares numberless values. In general we compare for effi-

ciency. Every day we have to make decisions based on some point of similarity between situations. Too often we decide by snap judgment when quantitative calculation is entirely practicable.

Motivating, practical, and educative, these problems are unduly neglected by our textbook writers. It is recommended that a large number be gathered and worked by the class itself, not only for motivating purposes, but also because such problems give the best kind of practice in the recognition and classification of the elements of the characteristic formulas involved. It certainly appears reasonable to teach unequal ratios before we teach equal ratios in which one factor of the characteristic formula is eliminated without consciousness of the process.

Sound quantitative thinking is developed when such problems are worked out in class somewhat as follows:

Problem. Which is the more profitable for a farmer and how much: to sell his corn on December 1 at 46 cents per bushel or on May 1 at 52 cents a bushel?

In May? Yes, but are there not other things to be considered? Suppose that the farmer must borrow money at 6 per cent . . .

He is still ahead. But suppose the shrinkage is 10 per cent . . .

And the teacher introduces other factors, such as cost of storage, insurance, difference in cost of hauling, or working time available.

That most people lack training in the discovery of the quantities involved in everyday business transactions is only too apparent to the observer. One instance will be sufficient. People buy houses on the installment plan. Apparently everything is calculated: interest on the money, insurance, repairs, taxes. But how many consider depreciation? Experts consider that about twenty-five per cent of business failures are due to oversight of depreciation alone!

Two-situation problems involving the comparison of two equal ratios — the proportion problems

$$A/B = K, \text{ and } a/b = K; \text{ therefore, } A/B = a/b$$

When the characteristic formula method is used, teaching the solution of proportion problems is easy. The simplest device that can be used in class demonstration consists of writing some characteristic formula vertically instead of horizontally on the blackboard. We can show that there are three possible relations between the two situations:

$Q = q$	$Q \quad q$	$Q \quad q$
$\parallel \quad \parallel$	$\parallel \quad \parallel$	$\parallel \quad \parallel$
$R \quad r$	$R = r$	$R \quad r$
$\times \quad \times$	$\times \quad \times$	$\times \quad \times$
$U \quad u$	$U \quad u$	$U = u$

Problem 1. If 12 yards of cloth cost \$48, what is the cost of 7 yards?

The characteristic formula involved is the cost formula:

$$C = RN$$

Adult reasoning shows that $R = r$ is the implied relation. But we have a better way of finding the missing relation and the operations required when we arrange the formulas vertically. Forty-eight dollars is evidently the total cost in one situation while the question is about the total cost in the other situation. Twelve and seven are clearly like quantities in both situations. Since they are not rates, they must be represented by N and n .

$$\begin{array}{rcl} 48 = C & c = ? \\ \parallel & \parallel \\ R & r \\ \times & \times \\ 12 = N & n = 7 \end{array}$$

We notice immediately that R and r are neither described nor related in the statement or diagram. But the fact that the rates are equal is the foundation of the solution. This rate-equality is then the relation between the two situations. It must be found before we can solve the problem; that is, we must do the following:

$$\begin{array}{rcl} 48 = C & c = ? \\ \parallel & \parallel \\ \longrightarrow & \\ 48/12 = R & = r = 48/12 \\ \times & \times \\ 12 = N & n = 7 \end{array}$$

We then proceed to solve for R and r :

$$\begin{aligned} \text{or,} \quad & 48/12 = R = r = c/7 \\ & 48/12 = c/7 \end{aligned}$$

— and the child sees every step of the solution.

It has been found by actual classroom use that this method is far easier to learn than the traditional method, besides being applicable to problems not containing the proportion feature. The traditional method, consisting of rules to be applied in various cases, fails to function for the simple reason that the child fails to recognize the proportion feature in problems outside of the chapter on proportion or loses faith in it after trying to solve problems that cannot be solved by proportion methods. If he does recognize it, he must decide upon the particular rule needed. After he has left school and forgotten the rule, he is helpless.

Some proportion problems are stated in such a way that the solution is very difficult if the numbers given are used. We cannot relate the two situations directly. We give an example, of the percentage type.

Problem 2. In a mass of solder weighing 600 pounds, there are 180 pounds of tin, and the rest is lead. How much tin should be added so that 100 pounds of the new solder shall contain 34 pounds of tin?

The method of solution usually given in textbooks has been found to be incomprehensible to most chil-

dren. Too much reasoning is required to solve it as stated. In order to relate the two situations directly, we restate the problem using the pounds of lead because the relation between the two situations lies in the fact that the lead remains unchanged in weight.

Problem restated: In 600 pounds of solder there are 420 pounds of lead. How much tin should be added so that 100 pounds of the new solder shall contain 66 pounds (66 per cent) of lead?

The characteristic formula is per cent = rate $\times$ base, or $P = RB$. Let T be the unknown amount of tin added. Then, since 420 is 70 per cent of 600, and the lead remains unchanged,

$$\begin{array}{rcl} 420 & = & 420 \\ \parallel & & \parallel \\ .70 & & .66 \\ \times & & \times \\ 600 & (600 + T) & \end{array}$$

or,

$$.70 \times 600 = 420 = 420 = .66(600 + T)$$

Whence $T = 36.36$ pounds ¹

¹ By the unitary method this problem would be solved as follows: 420 pounds become 66 per cent of the whole mass, M , after the tin is added. Therefore,

$$1 \text{ per cent of } M = \frac{420}{66}$$

and

$$100 \text{ per cent of } M = \frac{420 \times 100}{66} = 636.36 = M$$

Then subtract the original mass, 600 pounds.

Some teachers and authors write down the proportion

$$\frac{70}{66} = \frac{600 + T}{600}$$

persuade the class that the equation is correct, and make the children try to imitate their feat. An examination of a pupil's scratch pad would show the following "system":

$$\frac{70}{66} = \frac{600}{420 + T} \quad \frac{70 + T}{66} = \frac{420}{600} \quad \frac{70}{420} = \frac{600 + T}{66}$$

and similar combinations. That is, the children try all possible combinations of the numbers given until they stumble on the right one.

"These children will not think!" the disgusted teacher then exclaims. The children are willing to think, but they have not any plan, they do not even know how to start. All that teachers or books show are only the final results of complicated mental processes of perception and elimination.

Problem 3. 147 yards of rope cost \$4.67; what is the cost of 43 yards?

The quantities mentioned in the problem are length and cost. This suggests that the characteristic formula involved is $C = RN$, total cost = cost per unit of length multiplied by the number of units of length.

The total cost, C , is \$4.67, while N , the number of units of length having the same cost, is 147.

The third quantity described in the problem, 43, cannot belong to the same situation; it is the same kind of a quantity as 147, a number of yards; it is not a rate, the factor required to complete the group in the formula. There must, then, be two groups of related quantities. One of the groups is

$$C = RN, \text{ where } C = 4.67, \text{ and } N = 147.$$

That is, we can write

$$4.67 = 147 R$$

R is not given, but is seen to be equal to $4.67/147$.

The second group is $c = rn$. The question, What is the cost? shows that the formula must be of the aspect $c = rn$; total cost = ? We are given $n = 43$; do we know anything about r ? Yes, it is implied to be the same or equal to R , which in turn is $= 4.67/147$, and we have:

$$c = rn = 43 \times (4.67/147)$$

After this, the child should be shown the relation between the two situations by means of the formulas in vertical positions.

$$\begin{array}{rcl} 4.67 = C & c = ? & \\ \parallel & \parallel & \\ R = r & & \\ \times & \times & \\ 147 = N & n = 43 & \end{array}$$

This diagram shows clearly the operations to be performed. We find R and transfer the value obtained to the right-hand situation. In symbolic form this would proceed as follows:

$$\frac{C}{N} = R = r = \frac{c}{n}$$

or,

$$\frac{4.67}{147} = R = r = \frac{c}{43}$$

This device has proved successful in making the plan of operation clear to some children after all other methods had failed. But we should not trust to it alone. The notion of relationship between quantities and the nature of that relationship should be presented from as many angles as possible. The teacher should also constantly remind the class that there are two kinds of relations in a two-situation problem:

- (1) The characteristic formula;
- (2) The relation between like quantities in the two formulas involved. Also that the latter kind is often implied.

We may arrive at the same solution in a different way.

We have found that $R = 4.67/147$. In the second part of the problem, $r = c/43$.

But the rate is the same for both. That is, $R = r$. Therefore:

$$\frac{c}{43} = \frac{4.67}{147}$$

This process partakes of the nature of a short cut, for the elimination is done mentally. It should be used only after the other methods are fairly familiar. If used too early, the children are likely to try to save still more time and invent methods of "setting up" two or three equations, selecting the right one by comparing the results.

We can see now that the solution of proportion problems is easily taught when approached from the characteristic formula point of view. Proportion problems are not essentially different from other two-situation problems, and there is little justification for the present method of teaching them as if they were in a class by themselves, using short-cut rules which completely ignore the idea of relationship.

Many proportion problems can be handled as double problems. Thus our problem could have been stated as follows:

1. One hundred and forty-seven yards of rope cost \$4.67. At what rate is it sold, or what is the cost of one yard?
2. If the cost of one yard of rope — the rate — is $4.67/147$, what is the cost of 43 yards?

And then the solution would not have been so difficult to a beginner. Such a method would also develop power in unitary analysis, the only short-cut device, as has previously been pointed out, which has value from the pedagogical and the practical point of view.

Two-situation problems not involving proportion and leading to linear equations

Most algebraic problems in elementary courses are of this type. We have already given some illustrative examples in Chapter VII. The following are added because of the interesting analysis required. The first illustrates the difficulty encountered sometimes in the determination of the proper quantity to use.

Problem 1. Ten hours after a man had started on a trip, a second man followed, gaining 3 miles an hour on the first. When the second had been gone 8 hours, he met a traveler going in the opposite direction and at the same rate as his own. The traveler told him that he had passed the first man 2 hours 24 minutes before. How long did it take the second man to overtake the first?

This is clearly a problem requiring the characteristic formula $D = RT$. But while the rates and the times are easily determined, the distance is rather puzzling. If we take the distance to be measured to the point where one man overtakes the other, we find that we have not enough relations to solve the problem. Again, if we try to use the point where the second man meets the traveler, the data prove insufficient. Finally, we select the third and least conspicuous distance:

Let $D = d$ be the distance to the place where the *first* man met the traveler.

It took the first man $18 - 2.4 = 15.6$ hours to reach that place.

It would take the second man $8 + 2.4$ hours to reach the same place.

$$\begin{array}{rcl}
 D & = & d \\
 \parallel & & \parallel \\
 R & = & r + 3 \\
 \times & & \times \\
 10.4 = T & & t = 15.6
 \end{array}$$

Solution:

$$RT = rt, \quad \text{or} \quad (r + 3) \times 10.4 = 15.6 r$$

Whence

$$r = 6$$

The rest of the problem is now easy.¹

The following problem, solved in about the same way in several well-known algebras, illustrates the traditional method of solution. Note that the plan of solution is kept dark. The reason for each step can be perceived only after a review. That the originator lacked clear perception of relationships and consequently used poor analytic methods will appear when the solution is compared with the solution through the function method following it.

Problem 2. A train, after traveling an hour from *A*

¹ It seems that this problem could properly be classified also as a three-situation problem — one situation for each man. (*Editor.*)

toward B , meets with an accident which detains it half an hour; after which it proceeds at four fifths of its usual rate, and arrives an hour and a quarter late. If the accident had happened thirty miles farther on, the train would have been only one hour late. Find the usual rate of the train.

Let y = the number of miles from A to B ;
and $5x$ = the number of miles the train travels per hour.

(How would a beginner ever think to use this convenient number?)

Then $4x$ = the rate of the train after the accident.

Then $y - 5x$ = the number of miles the train has to go after the accident.

Hence

$$\frac{y - 5x}{5x} = \text{the number of hours required usually;}$$

and

$$\frac{y - 5x}{4x} = \text{the number of hours actually required.}$$

Therefore,

$$\frac{y - 5x}{4x} - \frac{y - 5x}{5x} = \text{the loss of running time in hours.}$$

But since the train was detained a half-hour and arrived one and a quarter hours late, the *running time* was three quarters of an hour more than usual.

That is:

$$3/4 = \text{loss in hours of running time.}$$

Therefore,

$$\frac{y - 5x}{4x} - \frac{y - 5x}{5x} = \frac{3}{4} \quad (1)$$

If the accident had happened thirty miles farther on, the remainder of the journey would have been $y - (5x + 30)$ miles, and the loss in running time would have been half an hour.

Therefore,

$$\frac{y - (5x + 30)}{4x} - \frac{y - (5x + 30)}{5x} = \frac{1}{2} \quad (2)$$

From the solution of (1) and (2), $x = 6$ and $5x = 30$.

This formidable problem was assigned to a class trained in "husking" problems and solving them by the function method. Half of the class brought in the following solution:

The following statement can be made: The difference in time between running thirty miles at schedule rate and at four fifths schedule rate is one quarter of an hour.

$$\begin{array}{rcl} D = & d = 30 \\ \parallel & & \parallel \\ R = & \frac{4}{5} r \\ \times & & \times \\ T = & t + \frac{1}{4} \end{array}$$

Solution:

$$D/R = d/r + 1/4$$

Substituting the values of D , d , and R ,

$$150/4 r = 30/r + 1/4$$

Multiplying by $4 r$,

$$150 = 120 + r$$

Whence

$$r = 30$$

Many two-situation problems leading to simple simultaneous equations can be solved without formal analysis. Such problems can be solved by the traditional method of "reading the problem and writing down the equation" or "expressing all the unknowns in terms of one." Success with easy problems leads the inexperienced mathematician to believe that the method is general. One illustration will be sufficient.

Problem 3. The composition of a two-pound bar of solder is half lead and half tin, and its cost is \$1.32. If the cost of tin is five times that of lead, what is the cost of one pound of tin?

The relation diagram shows clearly the reason why the equation can be "written down after reading carefully."

$$\begin{array}{rcl}
 Q + q & = & 1.32 \\
 \parallel & & \parallel \\
 1 = N & & n = 1 \\
 \times & & \times \\
 U = 5u & &
 \end{array}$$

Two-situation problem leading to quadratic equations

Problem. A man bought some tables for \$100. If the number had been one more, the price for each would have been \$5 less. How many did he buy?

The characteristic formula used is: $C = RN$, total cost = cost of one $\times$ number of tables.

The quantities are all clearly described or related. We write the relation diagram:

$$\begin{array}{rcl}
 100 = C & = & c = 100 \\
 \parallel & & \parallel \\
 R & = & r - 5 \\
 \times & & \times \\
 N & = & n + 1
 \end{array}$$

We have

$$c = rn = 100, \quad \text{or} \quad r = 100/n \quad (1)$$

and

$$C = RN = (r - 5)(n + 1) = 100 \quad (2)$$

Substituting (1) in (2)

$$(100/n - 5)(n + 1) = 100$$

a quadratic equation. Solving it, we obtain $n = 4$.

We should note that we can solve for N , R , or r just as easily. These alternate methods of solution should be emphasized. The pupils should not get the impression that a problem can be solved only in one way.

Three-situation problems

Many problems belong to this type. Many problems usually found in the "simultaneous equation" chapter, the simple "work" and "cistern" problems, are found to be three-situation problems.

Problem 1. A dealer has two kinds of coffee worth 30 and 40 cents a pound respectively. How many pounds of each must be taken to make a mixture of 70 pounds worth 36 cents per pound?

We have here three situations: the two kinds of coffee and the mixture. Each situation is controlled by the characteristic formula $C = RN$; total cost = cost per pound $\times$ number of pounds.

The relating statements are:

1. The cost of the mixture = the sum of the costs of the different kinds.
2. The weight of the mixture = the sum of the weights of the different kinds.

Let M = cost of the mixture.

$$\begin{array}{rcccl}
 C & + & c & = & M \\
 \parallel & & \parallel & & \parallel \\
 30 = R & 40 = r & 36 & & \\
 \times & & \times & \times & \\
 N & + & n & = & 70
 \end{array}$$

or, more simply

$$C + c = 2520 \quad (1)$$

$$\parallel \qquad \parallel$$

$$30 \qquad 40$$

$$\times \qquad \times$$

$$N + n = 70 \quad (2)$$

And we see clearly how the two equations

$$30 N + 40 n = 2520$$

and

$$N + n = 70$$

are obtained.

Problem 2. Two men together can do a piece of work in 4.8 days. One man requires 4 days less than the other to do the work alone. Find the number of days each requires alone.

This "work" problem contains three situations: One man doing the work alone, the other man doing the work alone, and the two men working together. Each situation is described by the characteristic formula, $A = RT$: amount of work = rate of work $\times$ time.

We can make the following relating statements:

1. The total amount, Q = the sum of the amounts.
2. The two men together can do it in 4.8 days.
3. One requires 4 days less than the other:

$$T = t + 4 \quad (1)$$

$$\begin{array}{ccc} Q = A & + & a \\ \parallel & & \parallel \end{array} \quad (2)$$

$$\begin{array}{ccc} \frac{Q}{4.8} = R_1 & R = \frac{Q}{T} & r = \frac{Q}{t} \\ \times & \times & \times \\ 4.8 & 4.8 & 4.8 \end{array}$$

Substituting the values of Q , A , and a obtained from the vertical equations into (2), we have,

$$4.8 R_1 = 4.8 R + 4.8 r$$

Dividing by 4.8, and substituting the values obtained from the third line of equations,

$$\frac{Q}{4.8} = \frac{Q}{T} + \frac{Q}{t}$$

Dividing by Q , and substituting the value $T = t + 4$, given in (1), we have:

$$\frac{1}{4.8} = \frac{1}{t+4} + \frac{1}{t}$$

Which is the form that some textbook writers expect the student to "write down from a careful study of the relations between the parts."

Reducing the last result leads to the quadratic equation:

$$4.8 t + 4.8 (t + 4) = t (t + 4)$$

Solving, we find $t = 8$, and hence $T = 12$.

Four-situation problems

We meet here two types: The strictly four-related-situation problem and the pair of two-situation problems. In the latter type the relation diagram is built in two sections.

Problem 1. Two men start from opposite shores of a lake to row directly across. Their rates of rowing are uniform, but not the same. They meet 720 rods from the left-hand shore, then each rows to the starting-point of the other, rests five minutes, and rows back again. This time they meet 440 rods from the right-hand shore. Find the distance across the lake.

It is clear that the five-minute statement is superfluous, as the time element involved is the period during which they row. We shall therefore neglect it. The implied relations between the two sets of situations are: x , the distance across, is the same, and the rates remain unchanged, in both sets.

Situations 1 and 2:

One man rows 720 rods; the other man rows the rest of the distance. The length of time is the same in both cases.

$$\begin{array}{rcl}
 720 = D & d = x - 720 & (1) \\
 \parallel & \parallel & \\
 R & r & \\
 \times & \times & \\
 T = t & &
 \end{array}$$

We can write the last equation,

$$\frac{D}{R} = T = t = \frac{d}{r}$$

or

$$\frac{r}{R} = \frac{d}{D}$$

Substituting the values of (1) in the last equation:

$$\frac{r}{R} = \frac{x - 720}{720} \quad (2)$$

Situations 3 and 4:

The values of D , T , d , and t are different from the corresponding values in Situations 1 and 2, but x , R , and r are unchanged.

$$\begin{array}{rcl} x - 720 + 440 = D' & d' = x - 440 + 720 \\ \parallel & \parallel \\ R & r \\ \times & \times \\ T' = t' \end{array}$$

Solving the equations in the way shown above:

$$\frac{r}{R} = \frac{x + 280}{x - 280} \quad (3)$$

Comparing (2) and (3):

$$\frac{x + 280}{x - 280} = \frac{x - 720}{720}$$

Clearing of fractions:

$$720x + 720 \times 280 = x^2 - 1000x + 720 \times 280$$

Whence $x = 1720$ rods.

The above method of eliminating like quantities in pairs can be applied to many problems of the same type.

This problem is interesting from another point of view. It can be solved arithmetically as follows:

1. The sum of the distances that the men row is three times the distance across. (Each man goes across once; then they meet.)
2. The first man rowed 720 rods of the first crossing.
3. Therefore he rowed 720 rods of each of the three hypothetical crossings, or 720×3 rods. But he lacked 440 rods of rowing twice across.

Therefore $2x - 440 = 720 \times 3$, or $x = 1720$.

This problem was offered for solution to a number of mathematics teachers. Some solved it by roundabout algebraic methods, and some did not solve it at all. It was also proposed to persons who had never studied symbol juggling. Some solved it by arithmetic methods, and some did not solve it at all. The remarkable thing is that the arithmetic solution requires a very keen perception of relationship among quantities, acquired in some mysterious fashion without formal training.

But such special solutions are not always possible, and the four-situation problem may lead, even in the case of our simple characteristic formulas, to an equation of the third degree, just as the two-situation problem often leads to an equation of the second degree.

The mathematician is then in his element, while the untrained man is helpless. While such problems are not handled in elementary mathematics, we shall exhibit one for the purpose of demonstrating the power of our method of attack.

Problem 2. An army column is twenty-five miles long and is marched twenty-five miles. At the same time that the column started to march, a courier started from the rear and rode forward to the head of the army, then immediately returned to the rear. He found that the rear was occupying the same ground as that occupied by the head of the army at the beginning of the march. Find how many miles the courier traveled.

Let x = distance that the army marches before the courier arrives at the head; then, while the army marches x miles, the courier rides $x + 25$ miles.

We plan to eliminate the rates as in the preceding problem.

Situations 1 and 2 :

$$\begin{array}{rcl} x = D & d = 25 + x & (1) \text{ and } (2) \\ \parallel & \parallel & \\ R & r & \\ \times & \times & \\ T = t & & \end{array}$$

Dividing (1) by (2),

$$\frac{D}{d} = \frac{RT}{rt} = \frac{x}{25 + x} = \frac{R}{r} \quad (3)$$

Situations 3 and 4 :

$$\begin{array}{rcl}
 25 = D & d = 25 + 2x & (4) \text{ and } (5) \\
 \parallel & \parallel & \\
 R & r & \\
 \times & \times & \\
 T = t & &
 \end{array}$$

Manipulating the equations as above,

$$\frac{R}{r} = \frac{25}{25 + 2x} \quad (6)$$

Since the rates remain unchanged in the various situations, we can compare equations (3) and (6),

$$\frac{x}{25 + x} = \frac{25}{25 + 2x}$$

Whence

$$x = 17.68 \text{ miles}$$

Problems leading to equations of the third degree

We give some examples of four-situation problems leading to equations of the third degree. It is hoped that these will be found useful in motivating the usually dreary study of such equations. It is also hoped that, when the technique of problem-building is more universally known, more problems of this type will be devised. So far as the author knows, these are the first problems leading to equations of the third degree ever devised for practical purposes. As was said be-

fore, such problems exist, but people solve them by guessing methods.

Problem 1. Three contractors bid on an excavating job. The equipment that A has would have enabled him to complete the job alone 14 weeks faster than B , and 21 weeks faster than C . All three were put to work at the same time and finished the job in ten weeks. How much did each get if the contract price was \$308,700?

This is recognized as a variation of the cistern problem. The characteristic formula is $A = RT$: amount of work = rate $\times$ time.

Let x be the number of weeks that it would have taken the first man alone, and Q = the total amount of work done.

$$\begin{array}{ccccccc}
 A_1 & + & A_2 & + & A_3 & = & Q \quad (1) \\
 \parallel & & \parallel & & \parallel & & \parallel \\
 \frac{Q}{x} = R_1 & & \frac{Q}{x+14} = R_2 & & \frac{Q}{x+21} = R_3 & & R \\
 \times & & \times & & \times & & \times \\
 10 = T_1 & = & T_2 & = & T_3 = T & &
 \end{array}$$

We have, by substitution in (1),

$$R_1 T_1 + R_2 T_2 + R_3 T_3 = Q$$

And, substituting the values of R_1 , R_2 , R_3 , and T ,

$$\frac{10 Q}{x} + \frac{10 Q}{x+14} + \frac{10 Q}{x+21} = Q$$

Dividing both sides by $10Q$,

$$\frac{1}{x} + \frac{1}{x+14} + \frac{1}{x+21} = \frac{1}{10}$$

Which reduces to:

$$x^3 + 5x^2 - 406x - 2940 = 0$$

From this, we obtain $x = 21$. The second man requires 35 weeks, and the third 42 weeks. The first man gets $10/21$ of the total amount, the second $10/35$, the third $10/42$.

If there had been four contractors in the problem, we should have had an equation of the fourth degree, and so on.

Problem 2. A man wishes to travel 135 miles in his automobile in a certain number of hours. After traveling 45 miles at the scheduled rate, he is delayed a half-hour. He then goes 5 miles per hour faster than before, traveling thus 70 miles more. Then he increases his speed 5 miles an hour for the remaining 20 miles, and arrives on schedule time. Find his original rate.

Since the distances are all given, we shall not use d for distance in the diagram.

We can make the following relating statements:

1. The sum of the distances is 135 miles.
2. The scheduled rate, R , equals the rate for the first 45 miles; the increased rate for 70 miles is

$R + 5$; for the 20 miles it is $R + 10$.

3. The total time is the traveling time + the delay.

$$135 = 45 + 70 + 20 \quad (1)$$

$$\begin{array}{ccccccc} \parallel & & \parallel & & \parallel & & \parallel \\ R & & R & & (R+5) & & (R+10) \end{array} \quad (2)$$

$$\begin{array}{ccccccc} \times & & \times & & \times & & \times \\ T & = & T_1 & + & T_2 & + & T_3 & + & 1/2 \end{array} \quad (3)$$

Solving each vertical equation for the T 's and substituting in (3),

$$\frac{135}{R} = \frac{45}{R} + \frac{70}{R+5} + \frac{20}{R+10} + \frac{1}{2}$$

Clearing of fractions:

$$R^3 + 15 R^2 - 1050 R - 9000 = 0$$

From this the value $R = 30$ is obtained.

It is clear that, if we had introduced one or more additional changes in the rate, we should have obtained equations of the fourth or higher degree. The relation diagram, however, would not have become any more complex.

Change of setting

The setting of a many-situation problem can be changed as easily as the setting of simpler problems. We give a few examples.

Example 1. Two competing merchants are selling suits of the same kind and at the same price. Each sells 12 suits. A cuts the price \$3 per suit and sells 8

suits at the new price. *B* cuts the price \$6 a suit and sells 18 at the new price. Each deposits the same amount in the bank as a result of the sale. What was the price before the cut?

Example 2. Twelve boxes of apples cost \$3 per box more than 8 boxes of pears. The apples and pears were exchanged for 12 boxes of peaches and 18 sacks of potatoes. The peaches cost as much as the apples, and the potatoes cost \$6 a sack less than a box of apples. What was the cost of a box of apples?

Example 3. Ten hours after a burglary was committed, a detective starts after the burglar, gaining 3 miles every hour on the former. After he had been gone 8 hours, he increased his speed 3 miles an hour. But at this time the burglar also increased his speed 3 miles an hour. It took the detective 12 hours to overtake him, after he had increased his speed. How far did they go?

These three examples can be solved by means of the same relation diagram, the only change being made in the interpretation of the characteristic formulas. We give the diagram for the last example.

$$\begin{array}{ccccccc}
 D & + & D_1 & = & d & + & d_1 \\
 \parallel & & \parallel & & \parallel & & \parallel \\
 3 + R & = & R_1 & R - 3 = r & & & r_1 = R \\
 \times & & \times & & \times & & \times \\
 8 = T & & T_1 = 12 & & t = 10 + 8 & & t_1 = 12
 \end{array}$$

Solving the vertical equations for the d 's and substituting in the first equation:

$$8 R + 12 R_1 = 18 r + 12 r_1$$

Substituting the values of R_1 , r , and r_1 ,

$$\begin{aligned} 8 R + 12 (R + 3) &= 18 (R - 3) + 12 R \\ 20 R + 36 &= 30 R - 54 \end{aligned}$$

Whence $R = 9$.

Interesting variations may be obtained by making one rate decrease while the other increases.

A further illustration

The following is an example of the class of problems which become possible of solution when the relation-diagram method is used. We may classify it as a problem having a very simple setting and very complex relationships. It should be especially noted that the symbolic manipulation is facilitated by the recognition of the fact that the quantities eliminated in pairs are of the same dimension, and therefore will bear a numerical ratio to each other. The reader will probably find it easier to build a diagram and translate the relations from the statement of the problem than to check the equations on the diagram below.

Problem. A man and a boy were hired to work on a certain job. They started at 8 A.M. and finished one half of the job the first day. The boy came alone at

8 A.M. the next day and worked for one third of the time that it would have taken the man to do the whole job alone. Then the man came and finished the job alone, working two hours later than he worked the first day, doing twice as much work as the boy did the first day. What proportion of the money should each get?

The characteristic formula is: Amount done = rate of working $\times$ time.

This is clearly a four-situation problem; two for each day. Let Q = one half of the job.

$$\begin{array}{c}
 \text{FIRST DAY} \\
 \text{Boy} \quad \text{Man} \\
 RT_2/2 = Q_3 + Q_4 = Q \\
 \parallel \quad \parallel \\
 r \quad R \\
 \times \quad \times \\
 t = T = T_1 + T_2 - 2 \quad (b)
 \end{array}$$

$$\begin{array}{c}
 \text{SECOND DAY} \\
 \text{Boy} \quad \text{Man} \\
 Q_1 + Q_2 = Q \\
 \parallel \quad \parallel \\
 r \quad R \\
 \times \quad \times \\
 2Q/3R = T_1 \quad T_2 = 2 - T_1 + T \quad (a)
 \end{array}$$

Solving (a) for Q ,

$$Q = 3RT_1/2$$

Also,

$$Q = Q_3 + Q_4$$

Therefore,

$$3 RT_1/2 = RT_2/2 + RT$$

or

$$T_2 + 2 T = 3 T_1$$

Substituting (b),

$$T_2 + 2 T_1 + 2 T_2 - 4 = 3 T_1$$

or

$$3 T_2 - 4 = T_1 \quad (1)$$

Also,

$$Q_1 + Q_2 = Q$$

By substitution,

$$\begin{aligned} rT_1 + RT_2 &= 3 RT_1/2 \\ rT_1 &= R (3 T_1/2 - T_2) \\ r/R &= (3 T_1 - 2 T) \div 2 T_1 \end{aligned}$$

Substituting (1):

$$\begin{aligned} \frac{r}{R} &= \frac{9 T_2 - 12 - 2 T_2}{6 T_2 - 8} \\ &= \frac{7 T_2 - 12}{6 T_2 - 8} \quad (2) \end{aligned}$$

Also

$$Q_3 = rt$$

or

$$RT_2/2 = r (T_1 + T_2 - 2)$$

or

$$\frac{r}{R} = \frac{T_2}{2(T_1 + T_2 - 2)} \quad \text{Substitute (1)}$$

$$= \frac{T_2}{8T_2 - 12} \quad (3)$$

Comparing (2) and (3)

$$\frac{T_2}{8T_2 - 12} = \frac{7T_2 - 12}{6T_2 - 8}$$

Which reduces to:

$$25T_2^2 - 86T_2 + 72 = 0, \text{ whence } T_2 = 2.$$

Substituting in (1), we find $T_1 = 2$, and $T = 2$.

Substituting in (2), we find $r/R = 1/2$, or $2r = R$, and since $Q = 1$ (one half of the job):

$$rt + RT = 1, \text{ or } 3r = 1/2.$$

$$r = 1/6, \text{ and } R = 1/3.$$

That is, the boy does one twelfth of the job per hour, and the man one sixth.

The difficulty of the symbolic solution will probably be better realized if the conventional symbols are used. Thus, let:

$$r = x, R = y, T_1 = w, T_2 = z, \text{ and } T = t.$$

Then the above equations can be written as follows:

$$\begin{aligned} 2/3 y &= w \\ xw + yz &= 1 \\ yz/2 + yt &= 1 \\ yz/2 &= xt \\ t &= w + z - 2. \end{aligned}$$

In this form there is not any hint as to the procedure to be followed. The conventional symbols for unknowns do not suggest that the ratios of some of the quantities are numerical.

Six- and eight-situation problems

The many-situation problem is often met. Bulky statements usually make such problems appear more difficult than they really are. The analyses are often easy, once the various situations are clearly understood, and the right symbolic notation is used. The following is an example of a six-situation problem, solved by eliminating rates in pairs.

Problem. A and B start to run a race to a certain post and back again. A, returning, meets B 90 yards from the post, and arrives at the starting-place 3 minutes before him. If, then, he returns immediately to meet B, he would meet him at one sixth of the distance between the post and the starting-point. Find the length of the course and the duration of the race.

Let the distance between the posts be $= x$.

Taking the first pair of situations, A, returning, meets B:

$$\begin{array}{rcl}
 x + 90 = D & d = x - 90 & (1) \\
 \parallel & \parallel & \\
 R & r & \\
 \times & \times & \\
 T = t & &
 \end{array}$$

Since $T = D/R$, we obtain by substitution in the T equation:

$$D/R = d/r, \text{ or } R/r = D/d.$$

Substituting this value in (1),

$$R/r = (x + 90) \div (x - 90) \quad (2)$$

Taking the last set of conditions:

If the distance between the two posts is x , then A runs $2x + x/6$, or $13x/6$, and B runs $2x - x/6$, or $11x/6$.

Then:

$$D/d = RT/rt = R/r = 13/11 \quad (3)$$

Comparing (2) and (3),

$$\frac{13}{11} = \frac{x + 90}{x - 90}$$

Whence $x = 1080$ yards.

The time can easily be found from Situation 2 and the ratio of the rates. We give the diagram,

$$\begin{array}{rcc} 1080 = D = d = 1080 & & \\ \parallel & \parallel & \\ R \div r = 13/11 & & \\ \times & \times & \\ T = t + 3 & & \end{array}$$

Whence $T = 16.5$.

This problem can be solved by arithmetic methods once the necessary relations are perceived. But

while such solutions can be perceived after the problem is solved, the relation-visualizing process is too difficult to be used in most cases. We give one solution:

A lead of 180 yards for one half the race makes 360 yards for the whole race.

But this is one sixth of the distance.

Therefore the whole distance is $360 \times 6 = 2160$ yards.

And the distance between the posts is 1080 yards.

Another six-situation problem.

Problem. Early in the morning mail accumulates in an office. During the working day it arrives at a uniform rate. At first the office practice was to put the entire force of 22 clerks to work sorting it at the beginning of the working day, the job being done in 20 minutes. Later the number of clerks was reduced to 12. These catch up with the incoming mail in one hour. As it is desired to reduce the office force still further, the manager wishes to know: (1) How many clerks will be needed to catch up with the incoming mail in 5 hours? (2) How many clerks will be needed to keep up with the mail the rest of the day?

The characteristic formula is: Amount = rate of work $\times$ time.

Let R = the rate of work of one clerk, and x = the number of clerks necessary to keep up with the incom-

ing mail. Then $x R$ is the rate at which the mail is arriving.

The basal relation is clearly:

Amount accumulated before working hours +
amount that comes in during working hours = amount
disposed of:

$$A + A_1 = A_2$$

It is clear that the rate of working is the rate of one clerk multiplied by the number of clerks.

$$\begin{array}{rcccl}
 A + A_1 & = & A_2 & & A + A_3 = A_4 \\
 \parallel & & \parallel & & \parallel & \parallel \\
 x R & & 12 R & & x R & 22 R \\
 \times & & \times & & \times & \times \\
 T_1 = T_2 = 1 & & & & T_3 = T_4 = 1/3
 \end{array}$$

Solution:

$$A + x R = 12 R \quad A + \frac{x R}{3} = \frac{22 R}{3} \quad (1) \text{ and } (2)$$

Solving (1) and (2) for A and comparing:

$$\begin{array}{l}
 12 - x = 22/3 - x/3 \\
 36 - 3x = 22 - x
 \end{array}$$

Whence $x = 7$.

That is, seven men can keep up with the mail. An obvious proportion solves the rest of the problem.

While this solution is clear and satisfies the beginner, the more advanced student would probably attack the

problem as follows, using relationships not so directly mentioned:

In both cases the accumulated amount is the same and hence must be removed in the same number of clerk-hours; that is, $(22 - x)$ clerks working for one third of an hour do the same work as $(12 - x)$ clerks working for one hour; or:

$$(22 - x) \times 1/3 = (12 - x) \times 1$$

The relation diagram shows the chain of reasoning:

$$\begin{array}{ccc} A = a & & \\ \parallel & \parallel & \\ (22 - x) = R & r = (12 - x) & \\ \times & \times & \\ 1/3 = T & t = 1 & \end{array}$$

But the teacher should not introduce this kind of short cut too early. While the child can understand the explanation, few beginners are capable of forming an abbreviated plan of solution.

We may change the setting of this problem to the following:

Example. The work on a contract is delayed by a storm. In order to get back to schedule time the contractor increases his force to 22 men for a period of 3 days. Later another storm of the same duration as the first forces him to employ 12 men for 9 days. If no delay had occurred, how many men would have been sufficient?

Or we may write it thus:

Example. Coal is on deck and coal is running on the deck from a chute at a uniform rate. Twelve men can clear the deck in an hour, while 22 men can clear it in 20 minutes. How many men can keep the deck clear after it has once been cleared?

An eight-situation problem leading to simultaneous cubics will finish our analysis of algebraic problems.

Problem. A launch sails across a still lake, then down a river having a current of 2 miles an hour, the distance down the river being equal to the distance across the lake. Without stopping, it then returns to its starting-point, the round trip taking 3 hours. Another launch, sailing faster by 2 miles an hour than the first, can make the round trip in 2 hours. What is the distance across the lake?

First Launch

$$\begin{array}{ccccccc}
 D & = & D & = & D & = & D \\
 \parallel & & \parallel & & \parallel & & \parallel \\
 R & & (R+2) & & (R-2) & & R \\
 \times & & \times & & \times & & \times \\
 T_1 & + & T_2 & + & T_3 & + & T_4 = 3
 \end{array}$$

Second Launch

$$\begin{array}{ccccccc}
 D & = & D & = & D & = & D \\
 \parallel & & \parallel & & \parallel & & \parallel \\
 (R+2) & & (R+4) & & R & & (R+2) \\
 \times & & \times & & \times & & \times \\
 t_1 & + & t_2 & + & t_3 & + & t_4 = 2
 \end{array}$$

Solution:

We obtain by substitution in the T and t equations,

$$\frac{2D}{R} + \frac{D}{R+2} + \frac{D}{R-2} = 3$$

and

$$\frac{2D}{R+2} + \frac{D}{R+4} + \frac{D}{R} = 2$$

or, clearing of fractions:

$$D \left[2(R+2)(R-2) + R(R-2) + R(R+2) \right] = 3R(R+2)(R-2)$$

and

$$D \left[2(R+4) + R(R-2) - (R+2)(R+4) \right] = 2R(R+2)(R+4)$$

Dividing the first by the second eliminates D .

$$\frac{2R^2 - 8 + R^2 - 2R + R^2 + 2R}{2R^2 + 8R + R^2 + 2R + R^2 + 6R + 8} = \frac{3(R-2)}{2(R+4)}$$

or

$$\frac{4R^2 - 8}{4R^2 + 16R + 8} = \frac{3R - 6}{2R + 8}$$

Whence

$$R^3 - 2R^2 - 14R + 4 = 0$$

One of the roots is 4.8 + miles an hour, and the distance can be found readily to be nearly 3.3 miles.

It can now be seen that our power of analysis is greater than our ability to solve. Even a beginner,

using the function method of analysis, can obtain from problems of the constant rate type equations so complex that the practiced mathematician's manipulative skill will be taxed to obtain a solution. And since analysis is comparatively easily learned, the necessity for mastering symbolic manipulation is again emphasized. This forms a natural basis for motivating formal drill.

CHAPTER IX

A JUSTIFICATION OF ALGEBRA IN SCHOOL AND IN LIFE

THE preceding chapters were concerned mainly with the development of sound and systematic methods of procedure. The material used was shown to be of secondary importance. The main aim is to teach the pupil to solve problems before teaching him any one kind of problem; to develop skills, habits, and attitudes rather than to impart specific knowledge. To be sure, problems with practical settings are used once in a while for the purpose of maintaining the indispensable connection with life and of stimulating the pupil to see that there are problems to be solved. But academic problems are the principal tools used.

On the other hand, the student will have to use his skill in the solution of life problems. On this every one agrees. But here, as in the case of the function concept and of other fundamentals, we find that, while every one speaks of life problems, no one defines them. Very few even try to establish the difficult connection between book and life. The following questions, then, are in order:

What are the problems that we meet in life?

What is their nature, their complexity; how do they present themselves? Just what skill, technique, and information is needed to solve them?

As in many other life situations, a thorough investigation of what is needed will suggest a definition of the trouble, a clear statement. This clear statement will suggest a method of procedure.

A fallacy

A widespread fallacy is that children will learn science and mathematics when confronted with real problems of their own. Let them feel the desire or need to do something, and they will study and reason in the attempt to satisfy that desire. So it is argued.

But we find in practice that nothing of the kind happens. The children are almost sure to resort to the crudest of means of their own invention unless they are trained beforehand to adopt technical methods. Instead of preparing for solution by means of a thorough study of the situation, they will start with some very vague plan, trusting to luck and inspiration to help them in the disposition of troublesome details.

This trait is not peculiar to children. Untrained and unskilled adults exhibit it. Even persons trained along given lines show it in other and unaccustomed conditions. With this trait in mind, let us examine the average citizen's need of algebra.

The need of algebra

The amount of training in algebra actually needed by the people at large has been a burning issue in edu-

cational circles for a long time. Eminent authorities have stated that all the algebra that can be assimilated should be taught to all the people capable of learning; just as great authorities have asserted that the amount actually needed by the masses is infinitesimally small. This divergence of opinion exists mainly because the exact nature of the material used in practice and of the material presented in class is rather vaguely understood.

The first group of experts has so far presented but few if any valid proofs that high-school algebra deserves an obligatory position on the curriculum. The members of the second group are in still worse position. Their strongest argument is founded on fallacious reasoning. They generally start from the right premise, "People use very little algebra," but arrive at the wrong conclusion, "Therefore they have very little need of it."

Inadequacy of usual research

The usual method of analyzing the situation by examining the nature, complexity, and frequency of problems met by the average person will always yield inconclusive results because:

1. The investigator cannot possibly visualize all of some one else's activities, opportunities, and needs. As a very large number of persons in each class of society have to be investigated and questioned before the name *scientific* can be applied to

the research, the difficulty of the task becomes apparent.

2. If the person questioned does not understand algebra well enough to use it, his opinion is of little value and the investigation of his activities and needs by the question method becomes almost impossible.
3. Seeking for specific or possible uses of algebra in trades or occupations is to a large extent a useless activity, unless the investigator is thoroughly familiar with all the conditions met at all times. Otherwise very few problems will be found.

Thirty years of industrial experience in shop and construction work, in constant contact with various classes of clerks, artisans, and laborers, have enabled the author to appreciate the needs, unexpressed wants, and dissatisfaction with the present methods of instruction felt by these people. Too many writers seem to have formed their opinions from reading one another's books and from interviewing a few educated foremen and employers.

A rational basis of investigation

The need of algebra in vocations or other life activities can best be appreciated after an examination of the results obtained by people who get along without it; how ineffectively they dispose of their material, their time, their energy, their earnings, their savings; how unwisely they act in various situations in which

algebraic notions and processes would be of assistance. It will be found that people do not use algebra simply because they do not know how to use it. They do not miss something that they never possessed. Not knowing what can be done through algebra, nor even what it is, people fail to perceive the need of it in their own lives.

They are much like savages who never miss the benefits and comforts of an undreamed-of civilization. When a savage finds himself in a situation where he would be benefited or his life saved through the use of the tools or drugs of civilization, he uses whatever makeshifts are available, prospers or fails, lives or dies, in blissful ignorance of the factors that really determine the issue. Similarly, our average citizen, when confronted by some situation requiring quantitative thinking and knowledge, calmly sidesteps the issue through the use of makeshifts, the guess and trial-and-error methods. If lucky in his guesses, he becomes a prominent business man firmly convinced of the uselessness of schooling. If unlucky, he is speedily removed from places of responsibility and his opinion is considered to be of no importance.

Needs more real than apparent

The need of "sound quantitative notions" is more real than apparent. In this respect it may be classified with a large number of other needs. Many people see no need of "sound hygienic notions," and have to be driven by law into taking the most elementary sanitary

precautions. The same may be said of sound notions about safety, fire hazards, citizenship, and so on. When we have not the slightest conception of the existence or use of a notion, we never miss it, and are likely to resent the attempts of well-meaning persons forcibly to impart information about it.

To the mathematically trained observer, who is also active in life affairs, the lack and need of quantitative training are everywhere apparent. By *quantitative training*, however, is not meant the ability to solve equations of any particular degree or complexity or to demonstrate theorems, but the ability to perceive the existence of quantities and their relations, accompanied by the ability to use the results of observation of these relationships in everyday activities.

But such perception and ability are seldom met and seldom developed in our schools because educators do not seem to be able to visualize the industrial or general life situations that lead to problems. Business men often comment on the fact that persons who study and teach only book mathematics frequently display an amazing lack of ability to recognize geometrical figures, relating lines and essential quantities in actual objects; that they even fail to perceive the existence of physical problems which are obvious to men more accustomed to handling their environments. How can we expect men so poorly equipped to plan mathematical courses providing training in recognition of essential measurements in practical work?

The cost of ignorance

The lack of clear objectives in teaching produces disastrous results. The words *inefficient*, *irresponsible*, *unreliable*, describe a large percentage of our population, lacking in sound quantitative reasoning ability, lacking the ability to relate, to associate, to put two and two together. Millions of people fail in their careers, millions waste their resources and the resources of the nation, millions waste their lives and health because the relations among the quantities surrounding them are unintelligible to them.

“But,” will object the reader, “how will the ability to solve simultaneous quadratics or to find the area of a cycloid help these people? Of what use are sines and cosines to a pick-and-shovel man? How will the ability to simplify an ‘apartment-house’ fraction help the busy housewife, the tired business man, or the movie actor?”

This brings us to the important part of our discussion. The kind of algebra taught at the present time is of little use to most people. While some transferred value may be claimed for it, and while some general development is undoubtedly obtained, these two ends hardly justify imposing the study on everybody. The important question before us is, then:

The key question

Are symbolic manipulation and the solution of a few book problems the whole of algebra for the average

man? Is the study of these two the study of algebra?

Since algebra is a key course, the fate of secondary mathematics depends upon the answer that educators will give to this question in the next few years. If they answer in the affirmative, then the widely prevalent tendency to remove mathematics from the secondary curriculum can only continue to gather strength. The people at large do not use secondary mathematics; they see no more use for it than they see for Greek, and will soon refuse to foot the bills for teaching it. Already the observant teacher notices a deterioration in the classroom attitude toward the subject. Bad news travels fast, and many schools enroll propagandists who tell their fellow students about more enlightened high schools where they have pitched "Math." overboard.

But if symbolic manipulation and academic problems are not the whole of algebra, then what is the rest? Or, rather, what is the whole of algebra? What did the people want when they demanded algebra in the high schools a few decades ago? An attempt will be made to answer this last question.

What the people want

People dimly perceived that algebra was the science of "figuring," of measuring indirectly. Calculating values has been a world's need ever since men began to design, sell, make, and use material; in short, to handle quantities. Mathematics was originally devised to

measure quantities expeditiously and accurately. Symbols were devised to expedite mathematical operations. Laws governing the use of these symbols were discovered. We all know the history of the development of the science and how the applications of mathematical methods revolutionized industry and enriched the few.

But how is the average citizen faring at present? His little stock of quantitative wisdom has not kept pace with the increased complexity of his business environment. If anything, it has fallen behind owing to reduction in number of his opportunities to learn from home environment. It is insufficient to enable him to share fairly in the benefits of civilization. People who have discovered his quantitative ignorance are taking advantage of him at every turn. Everything is cunningly devised to separate a "quantitatively ignorant" person from the results of his labor and to give him the minimum in return. Fashions are devised which make him discard perfectly good clothes, houses, machinery, and appliances long before they have outlived their usefulness. Small packages have replaced bulk goods. "Free" delivery and so-called "service" in numberless lines are indulged in by people who cannot afford them, but do not know that they cannot. Middlemen thrive whose sole excuse for considering themselves useful is that they are able to attach themselves leechlike to some unnecessary detail of distribution. Money that should be reserved for necessities or a "rainy day" is

wasted on fancied needs or illusory pleasures, the desire for which is created by advertising. Individual and national waste have replaced thrift. Our forests, our coal, our petroleum, the fertility of our lands, are going the same way as the buffaloes and wild pigeons went. Classes, blocs, and groups are trying to promote their selfish interests with blind disregard of the consequences to the rest of the people. Everywhere we notice poverty, discomfort, and discontent produced by the lack of quantitative thinking. Men seem to have lost their sense of relative cost, their sense of values.

What they are taught

But did they ever in their school years acquire enough of this sense to function under modern conditions? Have they ever learned, are our pupils learning now, to put two and two together? Are they getting now a training in associating, in relating quantities? No; they are merely learning to juggle symbols as fast as possible. They are taught (?) to express by means of algebraic symbols relations that they do not perceive and cannot express in plain English.

Our present aim seems to be to train them to work a hundred examples a day in case they should become assistants to persons who discover, formulate, and solve a hundred problems a day, and who leave the finishing details to their assistants. We develop specialists in the solution of examples; that is, our children

learn only a part of a mental process. We teach them how to steer and "step on the gas." A knowledge of the machine is not needed so long as they are willing to pay exorbitant prices for more or less fictitious service to keep the machine in running order.

The efficiency fad in elementary mathematics

A study of past conditions and methods is often illuminating. Our fathers did not try to cover as much ground as we do; what they did attempt they covered more thoroughly than we. Their arithmetics and algebras were of insignificant size when compared to our modern catalogues of rules and short cuts. They seem to have paid more attention to reasoning than to rules, to methods than to immediate results, to training than to information.

But this is an age of efficiency and of efficiency experts. The latter are people who can improve anything developed by our fathers simply by applying "scientific" methods to it. Unfortunately, these improvements are usually evanescent because some fundamental underlying principle has been overlooked. Thus we have had scientific management of labor which would have revolutionized industry if our workmen had but consented to be devitalized or brutalized in the course of a few years.

But children are more easily experimented upon than men. Also they stay in school only a few years; that is, educational mistakes, like medical mistakes,

are quickly hidden from view. If exposed by mischance, it is a simple matter to place the blame on teachers who did not follow instructions, or upon the natural perversity of youth; therefore, efficiency experts flourish in our educational system. They are applying scientific methods which "expose" the child to numberless bits of desultory information, and they make him go through the most efficient motions used in business or industry. The only thing he lacks is a clear understanding of what is acquired and of how to use it for any purpose whatsoever.

We may compare the modern school child looking for an education to a green athlete who wants to train for long-distance running. His efficiency expert has read that old-time athletes ran twenty miles every day when training, and his sense of efficiency rebels against such a waste of time and effort. Why not use modern methods to cover the distance? Why not use an automobile? The athlete could then cover twice the amount of ground and do it ten times as fast as the old-fashioned runner, besides learning how to run a machine. Tangible results will be obtained more quickly and with less effort!

Similarly, in mathematics rules and short cuts give quick and immediate results. Ten times as many problems are solved. A large amount of (book) ground is quickly covered. But these results are obtained at the expense of organization, thinking, and the formation of desirable habits of concentration and suspended judg-

ment. We save a good deal of time so far as obtaining answers to conventional questions is concerned, and we also save a deal of effort in slow, cumbersome, elementary thinking. But does this develop sound mental habits?

What the people need

People need the algebra of quantities and relationships before they can use the algebra of numbers and other symbols. Handling the symbols of thought is not synonymous with thinking.

But what is this mysterious algebra of quantities and relationships?

Casual observers are often puzzled by the efficient and successful way in which apparently ignorant and untrained men handle their business. According to all the canons of scholarship, they are incompetents, should fail, should be replaced by our college graduates. Instead of this they amass fortunes or give other evidence of ability and efficiency while the college graduates work for them in subordinate positions. A careful analysis will reveal at least one factor of their success: almost invariably they have sound quantitative notions and the ability to keep a problem in mind until it is solved to their entire satisfaction. Their reasoning shows that they perceive clearly the relations among quantities, although they cannot express them in symbolic language.

Two hundred and fifty high-school graduates were

given recently an examination containing this problem among others: If Sam can do a job in 2 days and Jim can do the same job in 3 days, how long would it take them to do the job if they worked together? One student answered 0 days, probably having attempted to subtract 3 from 2. Another, obtaining -1 by the same method, interpreted it "less than one day." Others, more fortunate in their selection, obtained 1 day, having subtracted 2 from 3. Quite a few obtained 5 days, and some deduced 6 days, the method being obvious. The most popular answer was $2\frac{1}{2}$ days — the "average length of time." Only six students gave the right answer. Of these six, how many solved the problem because they remembered the traditional method of solving "work" or "cistern" problems, and not through reasoning out the situation, unfortunately was not mentioned in the report, but it is fairly safe to assert that most did.

The same problem was proposed to a number of unschooled but successful men. While most of them were unable to give the correct answer, all the answers showed a recognition of the relationships involved and all their guesses approximated very closely the correct answer.

The students had juggled symbols in order to obtain meaningless answers, while the business men had kept the conditions of the problem in mind, gathered thoughts, arranged them, passed judgments, arrived at tentative conclusions, compared the conclusions with

the conditions, and reached final conclusions. They had used a rudimentary algebra of quantities and relations without symbols, crude and slow, to be sure, but effective enough for their needs. Most of them were entirely unaware of the fact that mathematical processes are in existence which would have enabled them to obtain correct answers in a fraction of the time used.

The arithmetic solution given by persons possessing sound habits of reasoning is as follows:

In one day A does one half of the work.

In one day B does one third of the work.

In one day A and B together do five sixths of the work.

Since five sixths of the work is done in five fifths of one day, one fifth of a day is required to do one sixth of the work, and six fifths of a day will be required to do the whole job.

But it is impossible to teach children to solve problems by such a method. A careful analysis of the solution shows that two quantities are inextricably mixed in the statements: the amount of work and the rate of doing the work. The teacher solving the problem vaguely perceives some quantities and relationships and uses terms which approximately translate his thoughts. He cannot explain the mental process. He can only show the finished product, the steps in reasoning. The pupil understands the reasoning and may even learn the steps for that particular problem, but is

absolutely incapable of solving some similar problem in which the relationships are different. Thus, the problem —

Two men together can do a piece of work in 4.8 days. One man requires 4 days less than the other to do the work alone. Find the number of days each requires alone —

cannot be solved by arithmetic methods.

Quantitative thinking

The efficient person, then, is he who has somehow acquired the ability to recognize situations and quantitative relations, to gather data toward a given end, to relate cause and effect, to form correct judgments. When such a person has a quantitative problem to solve, he systematically twists it around, analyzes it, keeps it in the focus of his attention, looking at it from different points of view, while the “stream of consciousness” continues to deliver new suggestions for examination, acceptance, or rejection. Through a process of weighing data, of trial-and-error, an approximately correct result is finally obtained. We may call this process arithmetic reasoning without numbers, or an algebraic method in which quantities are used instead of their symbols.

The trained mathematician would have solved the same problem correctly in a fraction of the time through the use of algebraic laws and symbols — if it

had been presented to him all formulated, with the irrelevant details removed, with all the conditions necessary, all ready for analysis. But trained mathematicians do not often solve the world's problems, usually because the problems are not formulated when they present themselves in the everyday processes of industry and commerce. They are solved by the man who discovers them, often without the formality of writing them down.

Problems in life

The last sentences give the key to the situation. Ninety-nine per cent of the world's problems are never formulated. Problems in industry and other activities seldom advertise themselves. All that there is to make you suspect that there is a problem of some sort somewhere around is the fact that things are not running smoothly, that your competitor is succeeding better than you are, or that your promotion is slow in coming. After the problem is solved, you realize that its difficulty lay in searching for, comparing, and selecting data; that is, in recognizing quantities. Symbolic solution is usually the least difficult part of the process, and approximate results usually answer the purpose.

This, then, is the reason why present-day methods fail to satisfy the public. We are not training children to recognize problems in their environment, to formulate them after recognizing them, to analyze them

after formulating them, to bring in related situations and quantities during the analysis. We do not even teach them what quantities are! All that we teach them to do is to take the results of some one else's observation, or, worse still, merely to take the results of analysis already translated into symbols, apply certain complicated rules and principles, and obtain some result meaningless to them. This is not the mathematics needed or wanted by the public, nor ever can be.

THE REMEDY

Specific definition of needs

The average citizen needs:

1. The ability to recognize problems in his environment;
2. The ability to recognize the quantitative features of these problems;
3. The habit of suspending judgment in quantitative matters, which habit leads to the practice of gathering all the factors bearing on a situation and of weighing their relative values;
4. The ability to state a problem properly;
5. The ability to reduce the solution of a problem to a single thought embracing a general plan;
6. A clear conception of the comparison process;
7. Skill in the translation process: in deciding when to multiply or divide, in selecting formulas, in writing equations;

8. Knowledge of quantities and units of measurement;
9. Manipulative skill and accuracy.

Without this last people naturally dislike and avoid written calculations. But manipulative skill and accuracy are of very little use without the other elements involved in quantitative thinking.

The nature of life problems

Earlier discussion in this book has outlined a method and technique by means of which quantitative thinking, conscious and systematic thinking about quantities and relationships, can be developed.

But in this chapter it has been shown that when the pupil has learned to solve problems of the types described, we cannot consider our task completed. We have not yet given him all that we should, for he is not yet ready to attack the problems of life.

It is not intended to examine here commercial problems dealing with money and goods. These quantities are usually well understood, and the problems involving them arise naturally and directly. Nor is it intended to examine the comparatively small number of complex problems met in specialized activities only, and hence of little interest to the general public. We are concerned rather with the large class of problems involving mensuration or the application of the prin-

ciples of mechanics. The increasing number of such problems found in elementary textbooks indicates that educators recognize their importance in civilized environment.

Experience has shown that training in the solution of these problems by means of formulas, rules, and type solutions seldom functions in later life. The principal reason for this is that an essential condition is usually overlooked.

A condition essential to the solution of practical problems

We are familiar with certain quantities, slightly acquainted with other quantities, and totally ignorant of still others. *We can perceive relationships between quantities only when these quantities are familiar.* The merchant is familiar with money, time, and percentage; the farmer can think quantitatively when live stock and crops are discussed; the chemist is at home among molecules; and so on. In these days of specialization, every trade and profession has its own problems involving characteristic quantities. These quantities are used so frequently that the relationships become as well known as the quantities. The result is that complex problems are often solved by persons apparently ignorant and untrained. But the process of acquiring such a familiarity is long-drawn-out and painful. We seek to accelerate it and make it less disagreeable when we teach algebra, that organized set of

principles and processes by means of which problems in general can be solved. We present problems selected from various phases of human activity. But are we not frequently overlooking the fundamental principle quoted above? Is it reasonable to expect that children can perceive relationships among quantities, when they have never perceived the quantities themselves?

Let an American teacher propose the following problem to his class: If a dozen apples cost sixpence, how many apples will you get for a crown? Will it be a sign of low mentality if a boy fails to solve this problem, or simply an indication that he does not know the relationship between the unfamiliar quantities, a penny and a crown?

The concept of the moment of a force as a unit quantity

Here is another illustration. Every teacher of physics knows that the principle of moments is very difficult to teach. A class may solve a large number of problems involving the law of the lever and other applications, yet simple problems requiring a clear perception of that quantity called *moment of a force*, the tendency of a force to cause rotation about a point, are not solved. Why not? Because the quantity called *moment of a force* is invariably eliminated in "writing down" the equation. We write $F \times L$, and the child perceives the quantities *force* and *lever-arm length*, but

does not perceive that quantity which tends to produce rotation about a point.¹

Ignorance of this quantity then obliges the use of a number of rules and formulas. We understand that the principle of moments is the unifying thought, yet we do not use the concept of a moment of a force as a single quantity when solving a problem. The characteristic relationship $M = FL$ is not learned, nor is the fundamental relationship $M = m$, when two situations are described.² Therefore, the formula $FL = fl$ is not understood. But when we put this formula in the form $\frac{F}{l} = \frac{f}{L}$ we go to the limit of absurdity by separating the quantities involved in each individual situation. Let us see how such a formula functions in life affairs.

Problems involving moments

A man going to market was asked to purchase a

¹ To a certain extent this is due to traditional methods in teaching physics. The unit moment is clearly the moment produced by a unit force at a unit distance from the point of rotation. But do we use the concept? It is interesting to note here that the unit of work is measured by a unit of force acting through a unit of distance; that is, it also equals $F \times L$. To avoid confusion, the scientist calls units of work *ergs*, but leaves the bewildered schoolboy to struggle with foot-pounds in measuring two different quantities. The former uses a word which means a unit quantity to him, while the latter has to use a composite word not suggestive of unit quantity and having to be interpreted in two different ways.

² This relationship is usually expressed as $M - m = 0$, or as $\Sigma M = 0$, which formulas are approved by logicians, but incomprehensible to too many students.

turkey for each of three neighbors. Upon his return it was found that the storekeeper had neglected to mark the weights separately. The receipt read: "To three turkeys, 37 lbs. @ 30¢ a pound, \$11.10." The turkeys being of different sizes and no scales being available, the problem was to determine the individual weights.

Note the distinctive feature of the situation. It is desired to know the individual weights. The only quantities apparently available for the indirect measurement of these weights are the total weight and the total cost. Can the relationship of these quantities to the individual costs be established? Obviously not without additional measurements of some kind, for the purpose of introducing additional quantities and relationships.

This problem was assigned to a class in physics after it had thoroughly(?) studied mechanics. No satisfactory solutions were offered. It was then suggested that the principle of moments could be used, and that it might be assumed that a foot-rule or measuring-tape was available. This hint did not produce any result. A mention of the law of levers did not help. Even the suggestion that the turkeys be balanced two at a time while tied to a balanced stick or to a stick of negligible weight did not prove sufficient. It was only when three equations containing the traditional x 's, y 's, and z 's were written that the class perceived that, since the equations could be solved, the problem could be solved. The class was then ready to "solve" problems in

which the numbers were different and in which hams or pumpkins were weighed instead of turkeys. But does this readiness show that they had really learned the principle of moments so that it would be available when needed?

The systematic mental process that should be used in the solution of this problem may be described somewhat as follows:

What scientific principle can be used in the solution? Since we want to find weights, we must use a principle involving weights. These are usually found by means of scales. Scales work on the principle that when moments are equal there is not any motion. Moments are measured by weights and lever-arms. Can we create a situation in which the lever-arms can be measured? We can, by balancing the turkeys two at a time. We should then have sets of measurements related by the fact that the sum of the weights is thirty-seven pounds.

The formal analysis of the problem would then be as follows:

We have a four-situation problem: turkeys A and B being balanced, then turkeys B and C being balanced.

The characteristic formula is $M = FL$; the tendency of a weight to cause turning is equal to the product of the weight by the lever-arm. The following statements can be made:

1. The sum of the weights A , B , and C is 37 pounds.

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2. The moments are equal one to the other (since there is no turning).

The relation diagram displays the situation:

$$\begin{array}{cccc}
 M_1 = M_2 & M_3 = M_4 & & \\
 \parallel & \parallel & \parallel & \parallel \\
 B & A + B + C = 37 & & \\
 \times & \times & \times & \times \\
 L_1 & L_2 & L_3 & L_4
 \end{array} \quad (1)$$

Solution:

$$M_1 = M_2 \text{ and } M_3 = M_4$$

or,

$$\begin{array}{l}
 BL_1 = AL_2 \text{ and } BL_3 = CL_4 \\
 A = \frac{BL_1}{L_2} \text{ and } C = \frac{BL_3}{L_4}
 \end{array}$$

Substituting these values of A and C in (1),

$$\frac{BL_1}{L_2} + B + \frac{BL_3}{L_4} = 37$$

And the lengths being known, the weights are quickly found.

A few more examples relating to the principles of mechanics are given to illustrate the need of sound quantitative notions in actual experience.

1. Three laborers were observed carrying straight timbers. One man supported one end, and the other two held a stick upon which the timber rested at a point one third of the total length, as measured from

the other end. This explanation was given by the laborers, and also by numerous other persons to whom the problem was later submitted: It stands to reason that the man at one end carries one third, and that the other two men carry two thirds, since the two thirds at their end, if cut off from the rest, would balance upon their stick!

2. A number of men were loading telegraph poles on flatcars. They were told not to load any car above fifteen tons. On the other hand, the freight rate was by the carload; so that underloading would increase the shipping cost. No scales were available nor were the weights of the poles known. This problem was cleverly solved thus by the foreman: He balanced each pole in order to find its center of gravity, which he marked. He then balanced each pole off center, using his own (known) weight on one side. By measuring the lever-arms, the approximate weight of each pole was then easily calculated.

3. A heavy engine was being installed. A substantial inclined runway was built, the plan being to roll the machine up to its foundation. No account was taken of the known weight of the engine nor of the pulling force available. The slope of the runway proved too steep, so, after wasting valuable time on futile efforts, the machine had to be backed off and the runway torn down and reconstructed correctly. The waste of time and material could have been avoided if the foreman had known the work equation $W = FL$,

the relationship $\text{work input} = \text{work output} + \text{friction}$, and had allowed twenty-five per cent for friction.

4. A small community drew its water supply from individual wells. After an epidemic of typhoid fever, forty householders combined resources and installed a pipe bringing water to a reservoir from a distant river. The size of the supply pipe was "estimated" by experimenting upon the amount delivered by a short section. Head and friction in long pipes and elbows were entirely overlooked factors. As soon as the reservoir was filled, the service of water to the households commenced. Although the supply pipe was running day and night, the reservoir was emptied in four weeks.

Sixteen more families joined the original group, a new "estimate" was made, and the first plant was duplicated. The two reservoirs were filled from the two pipes as before, before water was served to the households, but ten weeks later they were empty again.

Before putting down a third pipe of the same size as the other two, the householders wish to know whether it will provide enough additional water to insure a steady supply.

The problem may be restated as follows:

One full reservoir fed from one pipe serves 40 houses 4 weeks. Two reservoirs fed by two pipes serve 56 houses 10 weeks. The water flows into the reservoirs at a uniform rate. At what rate is the water flowing

into the reservoir and how much does the average household use?

Let the amount of water in one full reservoir = Q .

The amount incoming through one pipe in one week = G .

The number of reservoirs = N .

The number of households = n .

The amount that one household uses in one week = S .

The time in weeks = T .

We can make the relating statement: The amount of water in one reservoir (A_1) plus the amount that flows in (A_2) equals the amount used (A_3).

There will be two relation diagrams of the form:

$$\begin{array}{rcc}
 A_1 + A_2 = A_3 \\
 \parallel \quad \parallel \quad \parallel \\
 N \quad N \quad n \\
 \times \quad \times \quad \times \\
 Q \quad G \quad S \\
 \quad \times \quad \times \\
 \quad T = T
 \end{array}$$

Solution:

$$NQ + NGT = nST$$

In the first case,

$$Q + 4G = 160S$$

In the second case,

$$2Q + 20G = 560S$$

Solving these gives: $Q = 4G$

That is, the rate of inflow is one fourth of the reservoir capacity per week.

We also obtain: $Q = 80 S$

That is, one household uses one eightieth of the amount in one full reservoir each week (if no water flowed in). Hence, one supply pipe will provide enough water for twenty households.

$$G = 20 S$$

And one additional pipe will, therefore, be sufficient for the needs of the fifty-six households.

Mensuration formulas

Our discussion would not be complete without some consideration of mensuration, since most elementary textbooks contain mensuration problems.

The ability to visualize objects — that is, pictorial and spatial imagination — is very unequally developed in different children. Before assigning problems in mensuration, it is advisable to ascertain just how well the class can picture problem conditions. Presentations of the following type may be used:

Show the class a cube, the edge of which is about three inches long, and say:

“Notice that this cube is painted. Suppose that I cut it into three equal flat slabs, thus.” (Draw two straight chalk lines showing the cutting planes.) “I have cut the cube into slices one inch thick. Suppose

now that, keeping the slices together, I cut them thus." (Show two more chalk lines at right angles to the first ones.) "I now have some blocks one inch by one inch by three inches long. Keeping the blocks together in the form of the original cube, I cut them again, thus . . . I have now a number of small cubes. Some of them are painted on one face, some are painted on two faces, and so on. Now, without using pencil and paper, try to figure mentally how many cubes of each kind I have cut. Write down and hand in your result as soon as you are satisfied that you are right."

Repeat this explanation until every one understands the problem, and remove the cube from sight.

Some of the children will find as many as fifty-six small cubes, while others will be satisfied with about a fourth of that number. Many will overlook the cube that is not painted on any face, or stoutly maintain that they can see small cubes painted on five or six faces.

After this, instruct the children to imagine that a piece is cut from each vertex of the cube and to count the new vertices and the old and new edges and faces mentally. Tabulate the results, right and wrong, on the blackboard, and discuss with the class.

Teachers can obtain valuable information as to their pupils' ability from such tests, especially if the children are questioned about their mental processes.

It is a waste of time and effort to assign mensuration problems to be solved by rules and formulas without

first making sure that the children have some perception of the measurable features of the objects considered; that is, of the quantities involved. It can only lead to a process of blind substitution in some uncomprehended formula, followed usually by a manipulation of meaningless numbers, and producing a still more meaningless result. Thus the problem —

A billet of steel is a inches thick, $2a$ inches wide, and $4a$ inches long. What are its dimensions if its weight is 2000 pounds, and the weight of one cubic inch of steel is .283 pound?

— has been “solved” by an uncomprehending boy, with the result that the billet was found by him to be 200 feet thick, 400 feet wide, and 800 feet long!

Translation into mathematical terms

When compared with academic problems, life problems present the following distinctive features:

1. Unfamiliar quantities;
2. Interesting but attention-distracting details;
3. Names of objects frequently used instead of names of quantities.

The first two having already been discussed, we turn our attention to the third.

Industrial and commercial terms do not suggest directly the mathematical terms into which they must be translated before their relationships can be clearly seen. The solution of a life problem, then, usually in-

volves a process of abstraction and generalization additional to the processes required for academic problems. We have to translate the names of objects into mathematical words, names of quantities, which through frequent association have acquired the power to recall certain formulas and processes. In short, we translate into an academic setting. Thus before we analyze a pulley problem, we must see that the feature that can be analyzed by mathematical methods is the circle, and understand that every time we meet the word *pulley* in the problem it means *circle*. Such a process immediately removes distracting details, and brings to our assistance the area and circumference formulas and the notion that the parts are interrelated, so that a change in one produces corresponding changes in all.

Simple and obvious as this process of abstraction may seem to the habituated teacher, non-use of it is one of the main causes of the inability of students to apply to practical purposes what they have "learned," and hence one of the causes of their growing dissatisfaction and lack of interest as the course progresses. To the inexperienced teacher it seems a matter of course that, once the geometric forms have been learned by name and their properties described, their uses and applications should become evident. But in practice we find that nothing is self-evident save to him who sees it. In our teaching experience we find that children will read problems about familiar objects having circular or rectangular features without feeling

the slightest necessity for recalling the formulas concerning circles or rectangles. The association between the circle seen in the textbook as a simple mathematical figure occupying the field of attention alone, and the circle as one of the many measurable features of some ordinary object, has not been made. Hence the generalization that the properties of the circle become measurable features in circular objects is not made. For instance, children perceive pulleys, their diameters, and their circumferences, but apparently these perceptions remain unrelated, hence cannot be considered together.

We find that even formulas and rules apparently fail to form associations in children's minds. This is probably because rules do not state explicitly that definite relations exist among the quantities concerned, that some are functions of others. Worded as they are, rules do not cause perception of relative magnitude, hence pupils cannot form judgments of relative magnitude.

Absurd answers to problems are often the result of this "rule" method of teaching. To the question — What is the circumference of a circle having a diameter of three inches? — the pupil is likely to answer 9.4248 inches, 94.248 inches, or .94248 inches, if a mistake has been made in placing the decimal point. He recalls the rule, but has no perception of the relation between the magnitudes. The rule merely connects the circumference and diameter through a number; that is, it

does not cause the pupil to make any comparison between the two. For this reason it is to a large extent a useless activity to learn rules containing numbers as many textbooks give them, even though illustrative examples and problems immediately follow these rules.

Comparison of academic and practical problems

It will be instructive at this point to compare the formulations of one problem, first as given in academic textbooks, then as given in practical books. Such comparison leads to better methods of presentation.

Let us first take the academic setting.

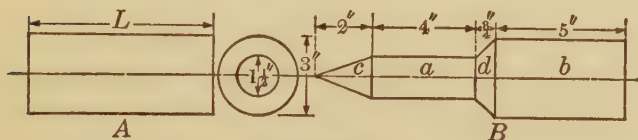


FIGURE 14

Find the length, L , of a cylinder, A , the volume of which is equivalent to that of the solid, B (Fig. 14). The parts a and b of the solid are cylinders, c is a cone, and d the frustum of a cone. The diameters of A and B are equal.

This statement contains all the technical words defining the quantities and relationships. It also contains a cue for formulating the hypothesis or plan of solution. But it does not contain any setting, anything to show that the student may ever have to solve such a problem in order to earn his living. It is only

one problem out of the many in the book that look just like it; hence it is not of particular interest.

In practical textbooks we should find the following setting:

B shows the dimensions of a steel pin forged from the billet *A*. What was the length of *A* before forging?

While this statement may be defended on the ground that it reproduces actual shop conditions, it is entirely unsuitable for beginners who are not very familiar with the mathematical terms into which the statement must be translated, and with the conventional methods used in showing shapes. It does not contain any clues to the solution. The essential relationship, that the volume remains unchanged, is difficult to perceive. Many students will get the impression that they are dealing with rectangular solids, or even with areas. This illustrates a defect commonly found in carelessly compiled books, one which needlessly aggravates the inherent difficulty of the problem.

Aside from this, an experienced boy would challenge the usefulness of the problem on the ground that the shop practice is to estimate the length needed before cutting, allowing about one inch for waste. The cost of the material wasted would be less than the cost of calculating the exact length needed. The same criticism will apply to many "practical" problems. It is feasible to remedy this objection by supplementing such statements with some element of high cost of

material, safety of operation, quantity production, and so on. This makes the problem more interesting without overloading the statement proper. Thus our problem might be presented as follows:

A blacksmith has to make 500 forgings of the shape shown in B (Fig. 14), from long round bars. These are cut into lengths, L , before forging. What should be the length, L , if no stock is to be wasted?

Supplementary statement:

The blacksmith needs the exact length because if the billets are too short they will be scrapped; if they are too long by one inch, 500 inches will be wasted. This means about 40 feet of the 3-inch bar. At 10 cents a pound, it would mean a waste of about \$75.

On looking over the last statement and comparing it with the other "practical" setting, we do not notice much difference so far as hints toward solution are concerned. The problem is more interesting, but it is more difficult than the academic problem. It must be simplified and translated into mathematical language before it can be solved. The latter alone has the power to recall automatically the formulas needed for the solution.

Translating problems into academic language may appear lengthy to practical-minded teachers and authors. To them it looks very simple to call certain quantities x and y , and to write down the equation at

once from the statement; then to explain why the equations are correct. But, while some beginners will be able to follow the reasoning, they will seldom learn the process. The ability to keep the end in view while mentally gathering the data is slow to develop.

After a number of problems have been worked out and the reasons for the various steps are well understood, the preliminary work can be dispensed with, at least on paper. But even then the pupil should be instructed to go through the whole process mentally every time he solves a problem. We cannot repeat too often that nothing can be considered learned until it can be consciously recalled for a definite purpose. It cannot be recalled consciously unless it has been associated consciously. Finally, this association can be formed only if the teacher or the book repeats a definite formula, showing the chain of associations every time a problem is presented.

Illustration of a chain of associations

In a fly-wheel problem the chain of associations would run somewhat as follows: We solve practical problems by translating them into mathematical language, because mathematical terms do not distract attention from the essential parts of the problem. The geometric name of the fly-wheel is the circle. When we hear the word *circle*, we must recall immediately that the various parts of the circle are related in a definite way, and that these relations are described in

formulas. What are the formulas of the circle? What are the formulas which show how the parts of the circle are related?

Once such a chain of thoughts is established, the solution of the practical problems found in books through the use of formulas becomes easy as well as valuable. We shall not, therefore, spend more time in discussing the subject, except to suggest that a table of the more commonly used formulas be made by the pupil. Such a table would be of great value in recalling automatically to him the fact that the elements of formulas are interrelated, and in showing how needed dimensions are found. Comparatively few formulas are needed; it is best to put them together.

We may summarize as follows:

The true function of the practical problem is first to create interest by keeping in touch with life; second, at the end of the course, to apply the knowledge and skill developed.

But so far as the actual learning process is concerned, the practical problem is worse than useless because: (1) the student cannot learn a systematic method of solution based on the study of quantity relationships; (2) he cannot acquire skill in symbolic manipulation, the mechanical skill which is necessary if the plan of solution is to be kept in mind instead of attention being diverted to the successive steps; (3) solution by rules and blind substitution is not quantitative thinking.

The "life" problems given in our textbooks often exhibit the worst features of both the academic and the practical problem. They are difficult, and they are not interesting; they do not train for life, and they do not train for mathematics. The golden mean is not in the compromise problem; it is in the sane combination of truly practical problems which aim to show the use of mathematics in difficult situations, and true academic problems presented with the sole aim of developing quantitative thinking and manipulative skill.

But the student should not be kept in the dark in this matter. He should understand the purpose of the academic drill. He should be allowed to try his powers of analysis and symbolic solution on the difficult life problem once in a while. Such a course would show him his deficiencies and needs better than any other method. In short, the student should be shown the end in view: the difficult life problem, and the means through which he can learn to solve it: the academic problem. He must firmly believe that there must be a judicious mixture of theory and practice, of how and why. He must build his habits on a real understanding of the subject.

The function concept in practical problems

The most important concept, the unifying idea of mathematics, is the function concept. But, while the notion of dependence, of variation, of functionality, is

useful in school mathematics, it is of little use when it is desired to establish fundamental relationships, and it is of still less use in practical life. In the latter case the notion that should be developed is not that quantities depend upon one another for their values, are related, vary together; but that quantities *can* be made to depend on one another, *can* be related, *can* be made to vary together.

This means that if we have some unknown quantity, the value of which must be found (the usual practical setting of a problem), we must start with the idea that there must be some quantity of the same kind somewhere in the environment, or one that can be brought into it. The unknown must be in some kind of a situation; quantity number two must be in some similar situation. The two situations must be or *can* be related, they must have something in common, or *can be made* to have something in common. The discovery or introduction of these relationships means the solution of the problem, for once found or established they can be stated in sentences and these sentences can be translated into equations.

The demonstration of such a process teaches the concept of a variable as a quantity that can be made to vary. An examination of the results of the changes produced in the physical environment by the variation of such a quantity would usually bring to light the related quantities, for these would also be found to vary.

Conclusion — A proposed sequence of topics

The process of the perception of quantities, of formulation, and of analysis described in these pages can be taught to the child of ordinary ability; it has been taught, and it is not difficult to teach once the objectives are clearly understood. The following sequence of topics has been successfully used in a year's course:

1. Academic problems, in which the quantities and their relationships are clearly and fully given.

Objective: To teach the recognition of quantities and the nature of relationships. To teach the statement of the relationships in plain English, then the translation of the relating sentences into equations, and the symbolic solution.

2. Academic problems, in which some quantities or relationships, or both, are implied, or in which the relationships are more complex.

Objective: To teach how to search for implied conditions.

3. Practical problems of type 1.

Objective: To teach translation into mathematical language. This step necessarily includes the study of technical terminology.

4. Practical problems of type 2.

Objective: To teach how to search for implied or related conditions.

5. Practical unformulated problems, stated as follows: How would you find . . . (give a single unknown quantity and nothing else).

Objective: To teach the student to *make* the relation, to search for significant data, and to formulate the problem.

6. Analysis of industrial, household, or civic problems stated as follows: Is this process efficient? What quantities enter as factors or can be made to enter as factors? Which can safely be varied? What would be the result of variation?

Objective: To develop the ability to discover problems in life work.

It is plain that the ability to handle problems of types 5 and 6 would be useful to most citizens. Those authorities who stated that people should be taught as much mathematics as they can assimilate had visions of problems of this type.

Those authorities who stated that the ability to solve quickly equations or formulas obtained from some one else's analysis is of slight value were also right. Riding in an automobile will never develop a distance runner!

Two conditions are essential for successful teaching:

First, the teacher must know what he is teaching; he must understand the objectives, the means, and the fundamental principles.

Second, the student must know why he has to learn the subject, and must have some conception of the objectives, the means, and the fundamental principles.

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